McGill University
Math 325B: Differential Equations
Notes for Lecture 3

Text: Section 2.6

In this lecture we will treat Bernoulli and homogeneous ODEs by means of a change of dependent variable to reduce the Bernoulli ODE to a linear one and the Homogeneous ODE to a separable one.

Bernoulli Equations. A Bernoulli ODE is a differential equation of the form

$$y' = p(x)y + q(x)y^n,$$

where $n \neq 0, 1$. Notice that if $n > 0$, the zero function $y = 0$ is a solution. So we assume that y is not the zero function. In order to transform this DE to a linear one we first divide both sides of the equation by y to get

$$\frac{y'}{y^n} = \frac{p(x)}{y^{n-1}} + q(x).$$

Setting $u = 1/y^{n-1}$ and using the fact that $\frac{du}{dx} = (1-n)y'/y^n$, the given ODE can be written

$$\frac{du}{dx} = (1-n)p(x)u + (1-n)q(x),$$

which is a linear equation.

Example. The ODE $y' = y - y^2$ is a Bernoulli equation. It is also a separable equation. We will solve it first as a Bernoulli equation and then as a separable equation and compare the two methods.

Dividing both sides of the DE by y^2 and setting $u = 1/y$, we get

$$\frac{du}{dx} + u = 1,$$

which is a linear equation with solution $u = 1 + Ce^{-x}$. This gives

$$y = \frac{1}{u} = \frac{1}{1 + Ce^{-x}}.$$

This, together with the solution $y = 0$, yields all solutions. Moreover, given a and $b \neq 0$, the equation

$$\frac{1}{1 + Ce^{-a}} = b$$

has the unique solution $C = (1-b)e^a/b$. It follows that the initial value problem

$$y' = y - y^2, \quad y(a) = b$$

has a unique solution for any a, b. For example, if $a = 0$, $b = 1/2$, the solution is

$$y = \frac{1}{1 + e^{-x}}$$

which increases from 0 to 1 as x increase from $-\infty$ to $+\infty$. It's graph has horizontal asymptotes $y = 0$ and $y - 1$. If $a = 0$, $b = -1$, then

$$y = \frac{1}{1 - 2e^{-x}}$$

This function decreases from 0 to $-\infty$ as x increases from $-\infty$ to $\log 2$ and decreases from $+\infty$ to 1 as x increases from $\log 2$ to $+\infty$. The graph of the function has a vertical asymptote at $x = \log 2$ and horizontal asymptotes $y = 0$ and $y = 1$.

To solve $y' = y - y^2$ by separation of variables, we write it in the form

$$\frac{y'}{y - y^2} = 1.$$

Since $\frac{1}{y-y^2} = \frac{1}{y} + \frac{1}{1-y}$, we get on integration of both sides with respect to x

$$\log|y| - \log|1 - y| = x + C_0$$

from which we get on exponentiating and dropping the absolute value signs

$$\frac{y}{1-y} = Ce^x.$$

with $C = \pm e^{C_0}$. On solving for y, we get

$$y = \frac{Ce^x}{1 + Ce^x} = \frac{C}{e^{-x} + C}.$$

Since $y = 0$ is a solution we can allow $C = 0$. The solution $y = 1$ is not part of this one parameter family of solutions.

Homogeneous Equations. A homogeneous ODE is one of the form

$$y' = F(\frac{y}{x}),$$

where F is some function. The function $f(x, y) = F(y/x)$ has the property

$$f(tx, ty) = f(x, y).$$

Such a function is said to be homogeneous of degree 0. Conversely, if $f(x, y)$ is homogeneous of degree 0, then

$$f(x, y) = f(1, y/x) = F(y/x)$$

with $F(x) = f(1, x)$. More generally, a function satisfying

$$f(tx, ty) = t^n f(x, y)$$

is said to be homogeneous of degree n. The ratio of two homogeneous functions of degree n is homogeneous of degree 0. For example, $ax + by$ is homogeneous of degree 0 while $ax^2 + bxy + cy^2$ is homogeneous of degree 2. It follows that

$$\frac{x - y}{x + y}, \quad \frac{xy}{x^2 + y^2}$$

are homogeneous of degree 0. Hence

$$y' = \frac{x - y}{x + y}, \quad y' = \frac{xy}{x^2 + y^2}$$

are homogeneous differential equations.

A homogeneous equation $y' = F(y/x)$ is solved by means of the change of variable $u = y/x$. Then $y = ux$ and $y' = xu' + u$ so that the differential equation becomes

$$xu' = F(u) - u$$

which is a separable equation.

Example To solve the homogeneous equation

$$y' = \frac{x-y}{x+y}$$

we make the change of variable $u = y/x$ and the DE becomes

$$xu' = \frac{1-u}{1+u} - u = \frac{1-2u-u^2}{1+u}.$$

Separating variables, we get

$$\frac{(1+u)u'}{1-2u-u^2} = \frac{1}{x}.$$

Integrating both sides with respect to x, we get

$$-\frac{1}{2}\log|1-2u-u^2| = \log|x| + C_0$$

or, equivalently, $(1-2u-u^2)x^2 = C$ with $C = \pm e^{-2C_0}$. Since $u = y/x$, we get

$$x^2 - 2xy - y^2 = C.$$

These curves are called **integral curves** of the ODE. They define the solutions implicitly. These curves are a family of hyperbolae whose asymptotes satisfy $x^2 - 2xy - y^2 = 0$. Solving for y, we find that the asymptotes are

$$y = -(1+\sqrt{2})x \text{ and } y = (\sqrt{2}-1)x.$$

One easily verifies that these are also solutions of the given DE. If $(a,b) \neq (0,0)$, there is a unique curve in the family (including $C = 0$) which passes trough the point (a,b).

Solving for y, we get

$$y = -x \pm \sqrt{2x^2 - C}.$$

The solution satisfying $y(1) = 1$ is $y = -x + \sqrt{2x^2+2}$ which is defined for all x. The solution satisfying $y(1) = 0$ is $y = -x + \sqrt{2x^2-1}$ which is defined on the interval $x \geq 1/\sqrt{2}$. However, the derivative of this function does not exist at $x = 1/\sqrt{2}$. This was to be expected since $y = -1/\sqrt{2}$ at this point and $(1/\sqrt{2}, -1/\sqrt{2})$ is a singular point of the differential equation.