McGill University
Math 325B: Differential Equations
Notes for Lecture 2

Text: Sections 2.1, 2.2,2.3

In this lecture we will treat linear and separable first order ODE's.

Linear Equations. The general first order linear ODE has the form

$$a_0(x)y' + a_1(x)y = b(x)$$

where $a_0(x)$, $a_(x)$, $b(x)$ are continuous functions of x on some interval I. To bring it to normal form $y' = f(x, y)$ we have to divide both sides of the equation by $a_0(x)$. This is possible only for those x where $a_0(x) \neq 0$. After possibly shrinking I we assume that $a_0(x) \neq 0$ on I. So our equation has the form (standard form)

$$y' + p(x)y = q(x)$$

with $p(x) = a_1(x)/a_0(x)$ and $q(x) = b(x)/a_0(x)$, both continous on I. Solving for y' we get the normal form for a linear first order ODE, namely

$$y' + p(x)y = q(x).$$

We now introduce the function,

$$\mu(x) = e^{\int p(x)dx}$$

It has the property $\mu'(x) = p(x)\mu(x)$ and $\mu(x) \neq 0$ for all x. Hence our differential equation is equivalent (has the same solutions) to the equation

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x).$$

Since the left hand side of this equation is the derivative of $\mu(x)y$, it can be written in the form

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x).$$

Integrating both sides, we get

$$\mu(x)y = \int \mu(x)q(x)d(x) + C$$

with C an arbitrary constant. Solving for y, we get

$$y = \frac{1}{\mu(x)} \int \mu(x)q(x)d(x) + \frac{C}{\mu(x)}$$

as the general solution for the general linear first order ODE

$$y' + p(x)y = q(x).$$

The function μ is called an **integrating factor** for the given linear ODE. Note that for any pair of scalars a, b with a in I, there is a unique scalar C such that $y(a) = b$. Geometrically, this means that the solution curves $y = y(x)$ are a family of non-intersecting curves which fill the region $I \times \mathbb{R}$.

Example 1: $y' + y = x$. This is a linear first order ODE in standard form with $p(x) = 1$, $q(x) = x$. The integrating factor is

$$\mu(x) = e^{\int dx} = e^x.$$

Hence, after multiplying both sides of our differential equation, we get

$$\frac{d}{dx}(e^x y) = xe^x$$

which, after integrating both sides, yields

$$e^x y = \int xe^x dx + C = xe^x - e^x + C.$$

Hence the general solution is $y = x - 1 + Ce^{-x}$. The solution satisfying the initial condition $y(0) = 1$ is $y = x - 1 + 2e^x$ and the solution satisfying $y(0) = a$ is $y = x - 1 + (a + 1)e^{-x}$.

Example 2: $xy' - 2y = x^3 \sin(x)$. We bring this linear first order equation to standard form by dividing by x. We get

$$y' + \frac{-2}{x}y = x^2 \sin(x).$$

The integrating factor is

$$\mu(x) = e^{\int -2dx/x} = e^{-2\ln|x|} = 1/x^2.$$

After multiplying our DE in standard form by $1/x^2$ and simplifying, we get

$$\frac{d}{dx}(y/x^2) = \sin(x)$$

from which $y/x^2 = -\cos(x) + C$ and hence $y = -x^2\cos(x) + Cx^2$, which is the general solution of the given ODE for $x \geq 0$ or $x \leq 0$ by continuity. Note that, if $a \neq 0$, there is a unique solution satisfying $y(a) = b$ for any constant b while all solutions satisfy the initial condition $y(0) = 0$. This non-uniqueness is due to the fact the DE in normal form is not well behaved at $x = 0$. However, $y = -x^2\cos(x) + Cx^2$ is not the general solution of the given ODE since different $C's$ are possible for $x \geq 0$ and $x \leq 0$ due to the fact that the one-sided derivatives at $x = 0$ are zero for all C. It is the general solution if y if y'' is required to exist at $x = 0$.

Separable Equations. The first order ODE $y' = f(x, y)$ is said to be separable if $f(x, y)$ can be expressed as a product of a function of x times a function of y. The DE then has the form $y' = g(x)h(y)$ and, dividing both sides by $h(y)$, it becomes

$$\frac{y'}{h(y)} = g(x).$$

Of course this is not valid for those solutions $y = \phi(x)$ at the points where $\phi(x) = 0$. Assuming the continuity of g and h, we can integrate both sides of the equation to get

$$\int \frac{y'}{h(y)} dx = \int g(x)dx + C.$$

This will in general give y implicitly in terms of x and one has to solve this implicit equation for y to get y explicitly as a function of x.

Example 1: $xy' = y$. Dividing both sides by xy, we get

$$\frac{dy}{y} = \frac{dx}{x}$$

which is equivalent to the given ODE when $xy \neq 0$. Solutions of the given ODE which cross the x or y-axis may be lost by this procedure. Integrating both sides of the second ODE, we get

$$\ln|y| = \ln|x| + C_0.$$

Exponentiating, we get

$$|y| = e^{\ln|y|} = e^{\ln|x|+C_0} = e^{\ln|x|}e^{C_0} = |x|e^{C_0}.$$

Removing the absolute values, we get $y = Cx$, where $C = \pm e^{C_0}$ which is also a solution of the given ODE when $x = 0$. Since $y = 0$ is a solution of the given ODE by inspection, we see that $y = Cx$ is a solution of the given ODE for any constant C. This is the family of straight lines passing through the origin with the exception of the $x - axis$. There is no solution of the given ODE satisfying $y(0) = b$ with $b \neq 0$. Any non-zero solution $y = y(x)$ must be $y = Cx$ with $C \neq 0$ for $x > 0$ and $x < 0$, possibly with different $C's$. However, different $C's$ is not possible as such a function would not be differentiable at $x = 0$. It follows that $y = Cx$ is the general solution of the given ODE.

Example 2: $(x+3)y' = y - 1$. By inspection, $y = 1$ is a solution. Dividing both sides of the given DE by $(y-1)(x+3)$ we get

$$\frac{y'}{y-1} = \frac{1}{x+3}.$$

Integrating both sides we get

$$\int \frac{y'}{y-1}dx = \int \frac{dx}{x+3} + C,$$

from which we get $\ln|y-1| = \ln|x+3| + A$. Thus $|y-1| = e^A|x+3|$ from which $y - 1 = \pm e^A(x+3)$. If we let $C = \pm e^A$, we get

$$y = 1 + C(x+3).$$

Since $y = 1$ was found to be a solution by inspection, we get the one-parameter family of solutions

$$y = 1 + C(x+3),$$

where C can be any scalar. This is the family of straight lines passing through the point $(-3, 1)$ with the exception of the line $x = -3$. Arguing as in the preceding example, the general solution of the given ODE is $y = 1 + C(x+3)$.