McGill University
Math 325B: Differential Equations
Notes for Lecture 17
Text: Ch. 4

In this lecture we will give a few techniques for solving certain linear differential equations with non-constant coefficients. We will restrict our attention primarily to second order equations. Some of the techniques can be extended to higher order equations.

Euler Equations. An important example of a non-constant linear DE is Euler's equation

$$x^2y'' + axy' + by = q(x),$$

where a, b are constants. This DE has a singular point at $x = 0$. For $x > 0$ this equation can be transformed into a constant coefficient DE by the change of independent variable $x = e^t$. This is most easily seen by noting that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = e^t\frac{dy}{dx} = xy'$$

so that $\frac{dy}{dx} = e^{-t}\frac{dy}{dt}$. In operator form, we have

$$\frac{d}{dt} = e^t\frac{d}{dx} = x\frac{d}{dx}.$$

If we set $D = \frac{d}{dt}$, we have $\frac{d}{dx} = e^{-t}D$ so that

$$\frac{d^2}{dx^2} = e^{-t}De^{-t}D = e^{-2t}e^tDe^{-t}D = e^{-2t}(D-1)D$$

so that $x^2y'' = D(D-1)$. By induction one easily proves that

$$\frac{d^n}{dx^n} = e^{-nt}D(D-1)\cdots(D-n+1)$$

so that $x^ny^{(n)} = D(D-1)\cdots(D-n+1)(y)$. Euler's equation then becomes

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = q(e^t),$$

a linear constant coefficient DE. Solving this for y as a function of t and then making the change of variable $t = \ln(x)$, we obtain the solution of Euler's equation for y as a function of x. To treat the case $x < 0$ we set $x = -e^t$ so that $t = \ln|x|$.

This method applies to the general n-th order Euler equation

$$x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_ny = q(x).$$

Example 1. Solve $x^2y'' + xy' + y = \ln(x)$. Making the change of variable $x = e^t$ we obtain

$$\frac{d^2y}{dt^2} + y = t$$

whose general solution is $y = A\cos(t) + B\sin(t) + t$. Hence

$$y = A\cos(\ln(x)) + B\sin(\ln(x)) + \ln(x)$$

is the general solution of the given DE.

Example 2. Solve $x^3y''' + 2x^2y'' + xy' - y = 0, \quad (x > 0)$. This is a third order Euler equation. Making the change of variable $x = e^t$, we get

$$(D(D-1)(D-2) + 2D(D-1) + D - 1)(y) = (D-1)(D^2+1)(y) = 0$$

which has the general solution $y = c_1e^t + c_2\sin(t) + c_3\cos(t)$. Hence

$$y = c_1x + c_2\sin(ln(x)) + c_3\cos(ln(x))$$

is the general solution of the given DE.

Exact Equations. The DE $p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$ is said to be exact if

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = \frac{d}{dx}(A(x)y' + B(x)).$$

In this case the given DE is reduced to solving the linear DE

$$A(x)y' + B(x) = \int q(x)dx + C$$

a linear first order DE. The exactness condition can be expressed in operator form as

$$p_0D^2 + p_1D + p_2 = D(AD + B).$$

Since $\frac{d}{dx}(A(x)y' + B(x)y) = A(x)y'' + (A'(x) + B(x))y' + B'(x)y$, the exactness condition holds if and only if $A(x), B(x)$ satisfy

$$A(x) = p_0(x), \quad B(x) = p_1(x) - p_0'(x), \quad B'(x) = p_2(x).$$

Since the last condition holds if and only if $p_1'(x) - p_0''(x) = p_2(x)$, we see that the given DE is exact if and only if

$$p_0'' - p_1' + p_2 = 0$$

in which case

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = \frac{d}{dx}(p_0(x)y' + (p_1(x) - p_0'(x))y).$$

The DE $p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$ can always be made exact by the multiplication by a non-zero function μ. Indeed, the DE

$$\mu p_0(x)y'' + \mu p_1(x)y' + \mu p_2(x)y = \mu q(x)$$

is exact iff $(\mu p_0)'' - (\mu p_1)' + (\mu p_2) = 0$ or, equivalently, if μ is a solution of the DE

$$p_0y'' + (2p_0' - p_1)y' + (p_0'' - p_1' + p_2)y = 0.$$

This DE is called the **adjoint** of the DE $p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$. This DE is equal to its adjoint, i.e. **self-adjoint**, iff $p_0' = p_2$. If $p_0' \neq 0$ it can always be made self-adjoint by multiplication by

$$\mu = e^{\int \frac{p_1 - p_0'}{p_0'}}.$$

Example 3. Solve the DE $xy'' + xy' + y = x, \quad (x > 0)$. This is an exact equation since the given DE can be written

$$\frac{d}{dx}(xy' + (x-1)y) = x.$$

Integrating both sides, we get

$$xy' + (x-1)y = x^2/2 + A$$

which is a linear DE. The solution of this DE is left as an exercise.

Example 4. The differential equation $y'' - xy - 2y = 0$ is not exact. Its adjoint is $y'' + xy' - y = 0$ which has $y = x$ as a solution. It follows that the DE

$$xy'' - x^2y - 2xy = 0$$

is exact; in fact, it can be written as $\frac{d}{dx}(xy' - (x^2+1)y) = 0$.

Reduction of Order

If y_1 is a non-zero solution of the second order DE

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$$

then $y = C(x)y_1$ is a solution of

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$$

if and only if

$$p_0(x)(C''(x)y_1 + 2C'(x)y_1' + C(x)y_1'') + p_1(x)(C'(x)y_1 + C(x)y_1') + p_2(x)C(x)y_1 = 0$$

or equivalently, on simplifying,

$$p_0y_1C''(x) + (p_0y_1' + p_1y_1)C'(x) = 0.$$

since $p_0y_1'' + p_1y_1' + p_2y_1 = 0$. This is a linear first order homogeneous DE for $C'(x)$. Note that to solve it we must work on an interval where $p_0(x)y_1(x) \neq 0$. However, the solution found can always be extended to the places where $p_0(x)y_1(x) = 0$ in a unique way by the fundamental theorem if the DE is non-singular at these points.

The above procedure can also be used to find a particular solution of the non-homogenous DE $p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$ from a non-zero solution of $p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$.

Example 4. Solve $y'' + xy' - y = 0$. Here $y = x$ is a solution so we try for a solution of the form $y = C(x)x$. Substituting in the given DE, we get

$$C''(x)x + 2C'(x) + x(C'(x)x + C(x)) - C(x)x = 0$$

which simplifies to

$$xC''(x) + (x^2+2)C'(x) = 0.$$

Solving this linear DE for $C'(x)$, we get

$$C'(x) = Ae^{-x^2/2}/x^2$$

so that

$$C(x) = A \int \frac{dx}{x^2 e^{x^2/2}} + B$$

Hence the general solution of the given DE is

$$y = A_1 x + A_2 x \int \frac{dx}{x^2 e^{x^2/2}}.$$

Example 5. Solve $y'' + xy' - y = x^3 e^x$. By the previous example, the general solution of the associated homogeneous equation is

$$y = A_1 x + A_2 x \int \frac{dx}{x^2 e^{x^2/2}}.$$

Substituting $y_p = xC(x)$ in the given DE we get

$$xC''(x) + (x^2 + 2)C'(x) = x^3 e^x.$$

Solving for $C'(x)$ we obtain $C'(x) = x^3 e^x$. This gives

$$C(x) = (x^3 - 3x^2 + 6x - 6)e^x + Bx.$$

We can therefore take

$$y_p = (x^4 - 3x^3 + 6x^2 - 6x)e^x$$

so that the general solution of the given DE is

$$y = A_1 x + A_2 x \int \frac{dx}{x^2 e^{x^2/2}} + (x^4 - 3x^3 + 6x^2 - 6x)e^x.$$