

McGill University  
Math 325A: Differential Equations  
Notes for Lecture 12  
Approximate Methods  
Text: Section 3.6

In this lecture we discuss methods for obtaining numerical solutions of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

at equally spaced points  $x_0, x_1, x_2, \dots, x_N = p, \dots$  where  $x_{n+1} - x_n = h > 0$  is called the step size. In general, the smaller the value of  $h$  the better the approximations will be but the number of steps required will be larger. We begin by integrating  $y' = f(x, y)$  between  $x_n$  and  $x_{n+1}$ . If  $y(x) = \phi(x)$ , this gives

$$\phi(x_{n+1}) = \phi(x_n) + \int_{x_n}^{x_{n+1}} f(t, \phi(t)) dt.$$

As a first estimate of the integrand we use the value of  $f(t, \phi(t))$  at the lower limit  $x_n$ , namely  $f(x_n, \phi(x_n))$ . Now, assuming that we have already found an estimate  $y_n$  for  $\phi(x_n)$ , we get the estimate

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for  $\phi(x_{n+1})$ . It can be shown that

$$|y_n - \phi(x_n)| \leq Ch,$$

where  $C$  is a constant which depends on  $h$ . This scheme is called **Euler's Method**.

The Euler method can be improved if we use the trapezoidal rule for estimating the above integral. Namely,

$$\int_a^b F(x) dx = \frac{1}{2}(F(a) + F(b))(b - a).$$

This leads to the estimate

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y_{n+1})).$$

If we now use the Euler approximation  $y_{n+1}$  to compute  $f(x_{n+1}, y_{n+1})$ , we get

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))).$$

This is known as the improved Euler method. It can be shown that

$$|y_n - \phi(x_n)| \leq Ch^2.$$

In general, if  $y_n$  is an approximation for  $\phi(x_n)$  such that

$$|y_n - \phi(x_n)| \leq Ch^p,$$

we say that the approximation is of order  $p$ . Thus the Euler method is first order and the improved Euler is second order.

One can obtain higher order approximations by using better approximations for  $F(t) = f(t, \phi(t))$  on the interval  $[x_n, x_{n+1}]$ . For example, the Taylor series approximation

$$F(t) = F(x_n) + F'(x_n)(t - x_n) + F''(x_n)(t - x_n)^2/2 + \cdots + F^{(p-1)}(x_n)(t - x_n)^{p-1}/(p-1)!$$

yields the approximation

$$y_{n+1} = y_n + hf_1(x_n, y_n) + \frac{h^2}{2}f_2(x_n, y_n) + \cdots + \frac{h^p}{p!}f_{p-1}(x_n, y_n),$$

where

$$f_k(x_n, y_n) = F^{(k-1)}(x_n) = \left(\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y}\right)^{(k-1)}f(x_n, y_n).$$

It can be shown that this approximation is of order  $p$ . However it is computationally intensive as one has to compute higher derivatives.

In the case  $p = 2$  this formula was simplified by Runge and Kutta to give the second order midpoint approximation

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right).$$

In the case  $p = 4$  they obtained the 4-th order approximation

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\ k_4 &= hf(x_n + h, y_n + k_3). \end{aligned}$$

Computationally, it is much simpler than the 4-th order Taylor series approximation from which it is derived.