McGill University
Math 325A: Differential Equations
Solutions for Test 1

1. Multiply DE by $x + e^y$ (problem when $x + e^y = 0$) and bring everything to one side to get $1 + (x + e^y)y' = 0$. With $M = 1$, $N = x + e^y$ we get

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = -1$$

which is a function of y only. Hence there is an integrating factor μ which is a function of y only which satisfies $\frac{d\mu}{dy} = \mu$. So e^y is an integrating factor. We therefore look for a function $F(x, y)$ with

$$\frac{\partial F}{\partial x} = e^y, \quad \frac{\partial F}{\partial y} = xe^y + e^{2y}.$$

Integrating the first with respect to x, we get $F(x, y) = xe^y + \phi(y)$. Differentiating this with respect to y and comparing with the second equation above, we get $\phi'(y) = e^{2y}$ from which $\phi(y) = e^{2y}/2$. Hence, $F(x, y) = xe^y + e^{2y}/2$. A solution to our DE therfore satisfies $xe^y + e^{2y}/2 = C$ and the initial condition $y(0) = 0$ yields $C = 1/2$. This gives

$$e^{2y} + 2xe^y - 1 = 0$$

a quadratic equation in e^y. Using the quadratic formula, we get

$$e^y = -x + \sqrt{x^2 + 1}$$

with the plus sign chosen to make the equation valid when $x = 0, y = 0$. Taking logs we get

$$y = \log(-x + \sqrt{x^2 + 1}).$$

The righthand side is defined for all x since $\sqrt{x^2 + 1} \geq |x|$ and is a solution since $x + e^y = \sqrt{x^2 + 1} \neq 0$.

Alternate Solution The DE can be written $\frac{dx}{dy} = -x - e^y$ which is a linear DE if we view x as a function of y. This yields $x = e^y/2 + Ce^{-y}$ and $C = -1/2$ when we put $x = y = 0$. The rest of the solution is as above.

2. This is a Ricatti equation which is also separable. We solve it as a Ricatti equation by letting $u = 1/y$ (check for possible solution $y = 0$) to get $xu' - u = -1$, a linear equation for u. This yields $u = 1 + Cx$ with C an arbitrary constant and hence $y = 1/(1 + Cx)$. Since $y = 0$ is also a solution, the general solution is $y = 0$ or $y = 1/(1 + Cx)$. Note that the second solution is never zero. If $x_0 \neq 0$, the initial condition $y(x_0) = x_0$ is satisfied by $y = 0$ if $y_0 = 0$ and by $y = 1/(1 + Cx)$ with $C = (1 - y_0)/x_0 y_0$ if $y_0 \neq 0$. If $x_0 = 0$, the initial $y(x_0) = y_0$ condition can only be satisfied if $y_0 = 0, 1$ since $x_0 y'(x_0) = y(x_0)^2 - y(x_0)$. If $y_0 = 0$ the only solution is $y = 0$ while if $y_0 = 1$ all the solutions $y = 1/(1 + Cx)$ satisfy $y(0) = 1$.

3. If x is the mass of salt in solution in the tank ant any time time we have

$$\frac{dx}{dt} = 3 - 3x/25$$

since the rate of input of salt is $6 \times .5 = 3$ and the rate of output is $6x/50 = 3x/25$. Solving this linear DE, we get

$$x = 25 + Ce^{-3t/25}.$$

Since $x(0) = 50 \times .1 = 5$, we get $C = -20$ which gives

$$x = 25 - 20e^{-3t/25}.$$

When the concentationof the salt in the tank is .3 we have $x = 15$ which yields $t = 25\log(2)/3 = 5.78$ (approx).

4. (a) Since $f(x,y) = x^2+y^2$ is continous along with its partial derivative $\frac{\partial f}{\partial y} = 2y$ on the rectangle $R : |x| \le a = 1, |y| \le b = 1$ and $|f(x,y)| \le M = 2$ on R, the fundamental existence and uniqueness theorem gives the existence of a unique solution on $|x| \le \min(a, b/M) = 1/2$.

   (b) We have $y' = x^2 + y^2,\ y'' = 2x + 2yy',\ y''' = 2 + 2(y')^2 + 2yy''$. Since $y(0) = 0$, we get $y'(0) = y''(0) = 0,\ y'''(0) = 2$. Hence

$$y(x) = y(0) + y'(0)x + y''(0)x^2/2 + y'''(0)/6 + \cdots = x^3/3 + \cdots .$$

   (c) If y_n is the n-th Picard iteration we have $y_0 = 0$ and

$$y_n = y_0 + \int_0^x (t^2 + y_{n-1}(t)^2)dt$$

so that

$$y_1 = \int_0^x t^2 dt = x^3/3$$

$$y_2 = \int_0^x (t^2 + t^6/9)dt = x^3/3 + x^7/63.$$

$$|y - y_n| \le \frac{M}{L}\frac{(LH)^{n+1}}{(n+1)!}e^{hL} \le e/(n+1)!$$

since $M = 2,\ L = 2,\ h = 1/2$.