McGill University
Math 325A: Differential Equations
Assignment 7 Solutions

1. The differential equation in normal form is

$$y'' + p(x)y' + q(x)y = y'' + \frac{1}{x}y' + \frac{1}{x} - \frac{1}{9x^2} = 0$$

so that $x = 0$ is a singular point. This point is a regular singular point since

$$xp(x) = 1, \quad x^2q(x) = -\frac{1}{9} + x$$

are analytic at $x = 0$. The indicial equation is $r(r-1)+r-1/9 = 0$ so that $r^2-1/9 = 0$, i.e., $r = \pm 1/3$. Using the method of Frobenius, we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Substituting this into the differential equation $x^2y'' + x^2p(x)y' + x^2q(x)y = 0$, we get

$$(r^2 - 1/9)a_0x^r + \sum_{n=1}^{\infty}(((n+r)^2 - 1/9)a_n + a_{n-1})x^{n+r} = 0.$$

In addition to $r = \pm 1/3$, we get the recursion equation

$$a_n = -\frac{a_{n-1}}{(n+r)^2 - 1/9} = -\frac{9a_{n-1}}{(3n+3r-1)(3n+3r+1)}$$

for $n \geq 1$. If $r = 1/3$, we have $a_n = -3a_{n-1}/n(3n+2)$ and

$$a_n = \frac{(-1)^n 3^n a_0}{n! 5 \cdot 8 \cdots (3n+2)}.$$

Taking $a_0 = 1$, we get the solution

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n a_0}{n! 5 \cdot 8 \cdots (3n+2)} x^n.$$

Similarly for $r = -1/3$, we get the solution

$$y_2 = x^{-1/3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n a_0}{n! 1 \cdot 4 \cdots (3n-2)} x^n.$$

The general solution is $y = Ay_1 + By_2$.

2. The differential equation in normal form is

$$y'' + p(x)y' + q(x)y = y'' + (\frac{1}{x} - 1)y' + \frac{1}{x} = 0$$

so that $x = 0$ is a singular point. This point is a regular singular point since

$$xp(x) = 1 - x, \quad x^2 q(x) = x$$

are analytic at $x = 0$. The indicial equation is $r(r-1) + r = 0$ so that $r^2 = 0$, i.e., $r = 0$. Using the method of Frobenius, we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Substituting this into the differential equation $x^2 y'' + x^2 p(x) y' + x^2 q(x) y = 0$, we get

$$r^2 a_0 x^r + \sum_{n=0}^{\infty} ((n+r)^2 a_n - (n+r-2) a_{n-1}) x^{n+r} = 0.$$

This yields the recursion equation

$$a_n = \frac{n+r-2}{(n+r)^2} a_{n-1}, \quad (n \geq 1).$$

Hence

$$a_n(r) = \frac{(r-1)r(r+1)\cdots(r+n-2)}{(r+1)^2(r+2)^2\cdots(r+n)^2} a_0.$$

Taking $r = 0$, $a_0 = 1$, we get the solution

$$y_1 = 1 - x - \sum_{n=2}^{\infty} \frac{(n-2)!}{(n!)^2} x^n.$$

To get a second solution we compute $a_n'(0)$. Using logarithmic differentiation, we get

$$a_n'(r) = a_n(r)\left(\frac{1}{r-1} + \frac{1}{r} + \cdots + \frac{1}{n+r-2} - \frac{2}{r+1} - \frac{2}{r+2} - \cdots - \frac{2}{r+n}\right).$$

Hence $a_1'(0) = 3a_0$ and $a_n'(r) = a_n(r)/r + a_n(r) b_n(r)$ for $n \geq 2$. Setting $r = 0$, we get for $n \geq 2$

$$a_n'(0) = \frac{(-1)\cdot 1 \cdot 2 \cdots \cdot (n-2)}{(n!)^2} a_0$$

from which $a_n = -(n-2)! a_0/(n!)^2$ for $n \geq 2$. Taking $a_0 = 1$, we get as second solution

$$y_2 = y_1 \ln(x) + 3x - \sum_{n=2}^{\infty} \frac{(n-2)!}{(n!)^2} x^n = y_1 \ln(x) + 4x - 1 + y_1.$$

The general solution is then $y = Ay_1 + B(y_1 \ln(x) + 4x - 1)$.