McGill University
Math 325A: Differential Equations
Assignment 7 Solutions

1. (a) $\mathcal{L}\{t\sin(t)\} = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{-2s}{(s^2+1)^2}$, $\mathcal{L}\{t\cos(t)\} = -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$,

$$\mathcal{L}\{t^2\sin(t)\} = -\frac{d}{ds}\frac{-2s}{(s^2+1)^2} = \frac{6}{(s^2+1)^2} - \frac{8}{(s^2+1)^3}, \quad \mathcal{L}\{t^2\cos(t)\} = -\frac{8s}{(s^2+1)^3} + \frac{2s}{(s^2+1)^2}.$$

(b) $\mathcal{L}^{-1}\{\frac{s}{(s^2+1)^3}\} = -t^2\cos(t)/8 + t\sin(t)/8$, $\mathcal{L}^{-1}\{\frac{1}{(s^2+1)^3}\} = 3\sin(t)/8 - 3t\cos(t)/8 - t^2\sin(t)/8$.

2. If $Y(s) = \mathcal{L}\{y(t)\}$, we have $(s^4-1)Y(s) - s^3 - s^2 + s + 1 = \frac{1}{s^2+1}$. Hence

$$\begin{aligned} Y(s) &= \frac{1}{(s^4-1)(s^2+1)} + \frac{((s+1)(s^2-1)}{s^4-1} = \frac{1}{(s^4-1)(s^2+1)} + \frac{((s+1)}{s^2+1} \\ &= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{1}{s+1} + \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{2}\frac{1}{(s^2+1)^2} + \frac{s}{s^2+1}. \end{aligned}$$

$$y(t) = e^t/8 - e^{-t}/8 + \sin(t)/2 + \cos(t) + t\cos(t)/4.$$

3. If $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$ we have

$$sX(s) - 1 = -2X(s) + 3Y(s), \quad sY(s) + 1 = X(s) - Y(s).$$

Hence $X(s) = (s-2)(s^2+3s-1)$, $\quad Y(s) = (-1-s)(s^2+3s-1)$ and

$$\begin{aligned} X(s) &= (\frac{7\sqrt{13}}{26} + 12)\frac{1}{s+(3+\sqrt{13})/2} + (\frac{-7\sqrt{13}}{26} + \frac{1}{2})\frac{1}{s-(\sqrt{13}-3)/2}, \\ Y(s) &= -(\frac{\sqrt{13}}{26} + 12)\frac{1}{s+(3+\sqrt{13})/2} + (\frac{\sqrt{13}}{26} + \frac{1}{2})\frac{1}{s-(\sqrt{13}-3)/2}, \\ x(t) &= (\frac{7\sqrt{13}}{26} + 12)e^{-(3+\sqrt{13})t/2} + (\frac{-7\sqrt{13}}{26} + \frac{1}{2})e^{(\sqrt{13}-3)t/2}, \\ y(t) &= -(\frac{\sqrt{13}}{26} + 12)e^{-(3+\sqrt{13})t/2} + (\frac{\sqrt{13}}{26} + \frac{1}{2})e^{(\sqrt{13}-3)t/2} \end{aligned}$$

4. We have $y'' + 3y' + 2y = 1 - 2u_1(t) + (\sin(t)+1)u_\pi(t)$. If $Y(s) = \mathcal{L}\{y(t)\}$

$$(s^2+3s+2)Y(s) = \frac{1}{2} - 2\frac{e^{-s}}{s} + e^{-\pi s}(\frac{-1}{s^2+1} + \frac{1}{s}),$$

since $\sin(t+\pi) = -\sin(t)$. Hence

$$Y(s) = \frac{1}{s(s+1)(s+2)} + e^{-s}\frac{-2}{s(s+1)(s+2)} + e^{-\pi s}(\frac{-1}{(s^2+1)(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}.$$

Since

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)},$$

$$\frac{1}{(s^2+1)(s+1)(s+2)} = \frac{1}{2(s+1)} - \frac{1}{5(s+2)} + \frac{1-3s}{10(s^2+1)},$$

we have

$$Y(s) = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} + (\frac{-1}{s} + \frac{2}{s+1} - \frac{1}{s+2})e^{-s}$$
$$+ (\frac{1}{2s} - \frac{3}{2(s+1)} + \frac{7}{10(s+2)} - \frac{1}{10(s^2+1)} - \frac{3s}{10(s^2+1)})e^{-\pi s}.$$

$$y(t) = \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} + (-1 + 2e^{1-t} - e^{2-2t})u_1(t)$$
$$+ (\frac{1}{2} - \frac{3e^{\pi-t}}{2} + \frac{7e^{2\pi-2t}}{10} + \frac{\sin(t)}{10} - \frac{3\cos(t)}{10})u_\pi(t).$$

Hence

$$y(t) = \begin{cases} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, & 0 \le t < 1, \\ -\frac{1}{2} + (2e-1)e^{-t} + (\frac{1}{2} - e^2)e^{-2t}, & 1 \le t < \pi, \\ (2e - 1 - \frac{3}{2}e^{\pi})e^{-t} + (\frac{1}{2} - e^2 + \frac{7}{10}e^{2\pi})e^{-2t} + \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t), & \pi \le t. \end{cases}$$