McGill University
Math 325A: Differential Equations
Assignment 5 Solutions

1. The differential equation in operator form is $(D+2)^2(y) = e^{-2x}\ln(x)$. Multiplying both sides by e^{2x} and using the fact that $D+2 = e^{-2x}De^{2x}$, we get $D^2(e^{2x}y) = \ln(x)$. Hence
$$e^{2x}y = \frac{1}{2}x^2\ln x - \frac{3}{4}x^2 + Ax + B$$
from which we get $y = Axe^{2x} + Be^{2x} + \frac{1}{2}x^2e^{2x}\ln x - \frac{3}{4}x^2e^{-2x}$. Variation of parameters could also have been used but the solution would have been longer.

2. The given functions $y1 = \cos x/\sqrt{x}$, $y_2 = \sin x/\sqrt{x}$ are linearly independent and hence a fundamental set of solutions for the DE
$$y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = 0.$$
We only have to find a particular y solution of the normalized DE
$$y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = \sqrt{x}.$$
Using variation of parameters there is a solution of the form $y = uy_1 + vy_2$ with
$$\begin{aligned} u'y_1 + v'y_2 &= 0, \\ u'y_1' + v'y_2' &= \sqrt{x}. \end{aligned}$$
By Crammer's Rule we have
$$u' = \frac{\begin{vmatrix} 0 & \frac{\sin x}{\sqrt{x}} \\ \sqrt{x} & \frac{-\sin x + 2x\sin x}{2x\sqrt{x}} \end{vmatrix}}{\begin{vmatrix} \frac{\cos x}{\sqrt{x}} & \frac{\sin x}{\sqrt{x}} \\ \frac{-\cos x - 2x\sin x}{2x\sqrt{x}} & \frac{-\sin x + 2x\sin x}{2x\sqrt{x}} \end{vmatrix}} = -x\sin x$$
$$v' = \frac{\begin{vmatrix} \frac{\cos x}{\sqrt{x}} & 0 \\ \frac{-\cos x - 2x\sin x}{2x\sqrt{x}} & \sqrt{x} \end{vmatrix}}{\begin{vmatrix} \frac{\cos x}{\sqrt{x}} & \frac{\sin x}{\sqrt{x}} \\ \frac{-\cos x - 2x\sin x}{2x\sqrt{x}} & \frac{-\sin x + 2x\sin x}{2x\sqrt{x}} \end{vmatrix}} = x\cos x$$
so that $u = x\cos x - \sin x$, $v = x\sin x + \cos x$ and
$$y = (x\cos x - \sin x)\frac{\cos x}{\sqrt{x}} + (x\sin x + \cos x)\frac{\sin x}{\sqrt{x}} = \sqrt{x}.$$
Hence the general solution of the DE $x^2y'' + xy' + (x^2 - 1/4)y = x^{5/2}$ is
$$y = A\frac{\cos x}{\sqrt{x}} + B\frac{\sin x}{\sqrt{x}} + \sqrt{x}.$$

3. This is an Euler equation. So we make the change of variable $x = e^t$. The given DE becomes
$$(D(D-1) + 3D + 1)(y) = e^{-t}/t$$
where $D = \frac{d}{dt}$. Since $D(D-1) + 3D + 1 = D^2 + 2D + 1 = (D+1)^2$, the given DE is
$$(D+1)^2(y) = e^{-t}/t.$$
Multiplying both sides by e^t and using $e^t(D+1) = De^t$, we get
$$D^2(e^t y) = 1/t$$
from which $e^t y = t\ln t + At + B$ and
$$y = Ate^{-t} + Be^{-t} + te^{-t}\ln t = A\frac{\ln x}{x} + \frac{B}{x} + \frac{\ln x}{x}\ln(\ln(x)),$$
the general solution of the given DE.

4. Using reduction of order, we look for a solution of the form $y = xv$. Then $y' = xv' + v$, $y'' = xv'' + 2v'$ and
$$(1-x^2)(xv'' + 2v') - 2x(xv' + v) + 2xv = 0$$
which simplifies to
$$v'' + \frac{2-4x^2}{x-x^3}v' = 0$$
which is a linear DE for v'. Hence
$$v' = e^{\int \frac{2-4x^2}{x^3-x}dx}.$$
Since
$$\frac{2-4x^2}{x^3-x} = \frac{-2}{x} + \frac{-1}{x+1} + \frac{1}{1-x},$$
we have
$$v' = 1/x^2(1-x)(x+1) = -\frac{1}{x} + \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}.$$
Hence $v = -\frac{1}{x} + \frac{1}{2}\ln\frac{1+x}{1-x}$ and the general solution of the given DE is
$$y = Ax + B(-1 + \frac{x}{2}\ln\frac{1+x}{1-x}.$$

5. This DE is exact and can be written in the form
$$\frac{d}{dx}(xy' + (x-1)y) = x$$
so that $xy' + (x-1)y = x^2/2 + C$. This is a linear DE. Normalizing, we get
$$y' + (1 - 1/x)y = x/2 + C/x.$$
An integrating factor for this equation is e^x/x.
$$\frac{d}{dx}(\frac{e^x}{x}y) = e^x/2 + Ce^x/x^2,$$
$$\frac{e^x}{x}y = e^x/2 + C\int\frac{e^x}{x^2}dx + D,$$
$$y = x/2 + Cxe^{-x}\int\frac{e^x}{x^2}dx + Dxe^{-x}.$$