McGill University
Math 325A: Differential Equations
Solutions to Assignment 4B

1. (a) Suppose that $M + Ny' = 0$ has an integrating factor u which is a function of $z = x + y$. Then $\frac{\partial(uM)}{\partial y} = \frac{\partial(uN)}{\partial x}$ gives

$$u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{\partial u}{\partial x} - N\frac{\partial u}{\partial y}.$$

By the chain rule we have

$$\frac{\partial u}{\partial x} = \frac{du}{dz}\frac{\partial z}{\partial x} = \frac{du}{dz}, \quad \frac{\partial u}{\partial y} = \frac{du}{dz}\frac{\partial z}{\partial y} = \frac{du}{dz},$$

so that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = \frac{-1}{u}\frac{du}{dz},$$

which is a function of z. Conversely, suppose that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = f(z),$$

with $z = x + y$. Now define $u = u(z)$ to be a solution of the linear DE $\frac{du}{dz} = -f(z)u$. Then

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = \frac{-1}{u}\frac{du}{dz},$$

which is equivalent to $\frac{\partial(uM)}{\partial y} = \frac{\partial(uN)}{/partialx}$, i.e., that u is an integrating factor of $M + Ny'$ which is a function of $z = x + y$ only.

(b) For the DE $x^2 + 2xy - y^2 + (y^2 + 2xy - x^2)y' = 0$ we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = \frac{2}{x + y} = \frac{2}{z}.$$

If we define

$$u = e^{\int -2dz/z} = e^{-2\ln z} = 1/z^2 = 1/(x + y)^2$$

then u is an integrating factor so that there is a function $F(x, y)$ with

$$\frac{\partial F}{\partial x} = uM = \frac{x^2 + 2xy - y^2}{(x + y)^2}, \quad \frac{\partial F}{\partial y} = uN = \frac{y^2 + 2xy - x^2}{(x + y)^2}.$$

Integrating the first DE partially with respect to x, we get

$$F(x, y) = \int (1 - \frac{2y^2}{(x + y)^2} dx = x + \frac{2y^2}{x + y} + \phi(y).$$

Differentiating this with respect to y and using the second DE, we get

$$\frac{y^2 + 2xy - x^2}{(x + y)^2} = \frac{\partial F}{\partial y} = \frac{2y^2 + 4xy}{(x + y)^2} + \phi'(y)$$

so that $\phi'(y) = -1$ and hence $\phi(y) = -y$ (up to a constant). Thus

$$F(x, y) = x + \frac{2y^2}{x + y} - y = \frac{x^2 + y^2}{x + y}.$$

Thus the general solution of the DE is $F(x, y) = C$ or $x+y = 0$ which is the only solution that was missed by the integrating factor method. The first solution is the familly of circles $x^2 + y^2 - Cx - Cy = 0$ passing through the origin and center on the line $y = x$. Through any point $\neq (0, 0)$ there passes a unique solution.

2. (a) The dependent variable y is missing from the DE $xy'' = y' + x$. Set $w = y'$ so that $w' = y''$. The DE becomes $xw' = w + x$ which is a linear DE with general solution $w = x\ln(x) + C_1x$. Thus $y' = x\ln(x) + C_1$ which gives

$$y = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C_1\frac{x^2}{2} + C_2 = \frac{x^2}{2}\ln(x) + A\frac{x^2}{2} + B$$

with A, B arbitrary constants.

(b) The independent variable x is missing DE $y(y-1) + y'^2 = 0$. Note that $y = C$ is a solution. We assume that $y \neq C$. Let $w = y'$. Then

$$y'' = \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = w\frac{dw}{dy}$$

so that the given DE becomes $y(y-1)\frac{dw}{dy} = -w$ after dividing by w which is not zero. Separating variables and integrating, we get

$$\int \frac{dw}{w} = -\int \frac{dy}{y(y-1)}$$

which gives $\ln|w| = \ln|y| - \ln|y-1| + C_1$. Taking exponentials, we get

$$w = \frac{Ay}{y-1}.$$

Since $w = y'$ we have a separable equation for y. Separating variables and integrating, we get $y - \ln|y| = Ax + B_1$. Taking exponentials, we get $e^y/y = Be^{Ax}$ with A arbitrary and $B \neq 0$ as an implicit definition of the non-constant solutions.

3. (a) The asociated homogeneous DE is $(D^3 - 3Dy + 2)(y) = 0$. Since

$$D^3 - 3D + 2 = (D-1)^2(D-2)$$

this DE has the general solution $y_h = (A + Bx)e^x + Ce^{2x}$. Since the RHS of the original DE is killed by $D - 1$, a particular solution y_p of it satisfies the DE

$$(D-1)^3(D-2) = 0$$

and so must be of the form $(A + Bx + Ex^2)e^x + Ce^{2x}$. Since we can delete the terms which are solutions of the homogeneous DE, we can take $y_p = Ex^2e^x$. Substituting this in the original DE, we find $E = 1/6$ so that the general solution is

$$y = y_h + y_p = (A + Bx)E^x + Ce^{2x} + x^2e^x/6.$$

(b) The associated homogeneous DE is $(D^4 - 2D^3 + 5D^2 - 8D + 4)(y) = 0$. Since

$$D^4 - 2D^3 + 5D^2 - 8D + 4 = (D-1)^2(D+4)$$

this DE has general solution $y_h = (A + Bx)e^x + E\sin(2x) + F\cos(2x)$. A particular solution y_p is a solution of the DE

$$(D^2+1)(D-1)^2(D^2+4)(y) = 0$$

so that there is a particular solution of the form $C_1\cos(x) + C_2\sin(x)$. Substituting in the original equation, we find $C_1 = 1/6$, $C_2 = 0$. Hence

$$y = y_h + y_p = (A + Bx)e^x + E\sin(2x) + F\cos(2x) + \frac{1}{6}\cos(x)$$

is the general solution.

4. (a)

$$W(\sin(x), \sin(2x), \sin(3x)) = \begin{vmatrix} \sin(x) & \sin(2x) & \sin(3x) \\ \cos(x) & 2\cos(2x) & 3\cos(3x) \\ -\sin(x) & -4\sin(2x) & -9\sin(3x) \end{vmatrix}$$

so that $W(\pi/2) = -16 \neq 0$. Hence $\sin(x), \sin(2x), \sin(3x)$ are linearly independent.

(b) The DE $(D^2+1)(D^2+4)(D^2+9)(y) = 0$ has basis

$$\sin(x), \sin(2x)\sin(3x)\cos(x), \cos(2x)\cos(3x)$$

and the given functions are part of it.

(c) Since the Wronskian of the given functions is zero at $x = 0$ it cannot be a fundamental set for a necessarily third order linear DE.