McGill University
Math 325A: Differential Equations
Solutions Sheet for Assignment 2B: due Thursday, September 21, 2000

1. Separating variables and integrating we get

$$\int \frac{yy'dx}{(y^2-1)^{4/3}} = \frac{x^2}{2} + C_1$$

from which, on making the change of variables $u = y^2$, we get

$$\frac{1}{2}\int (u-1)^{-4/3}du = \frac{x^2}{2} + C_1.$$

Integrating and simplifying, we get

$$(u-1)^{-1/3} = C - x^2/3 \quad \text{with} \quad C = -2C_1/3.$$

Hence $(y^2-1)^{-1/3} = C - x^2/3$. Then $y(0) = b$ gives $C = (b^2-1)^{-1/3}$. Since $b > 0$ we must have

$$y = \sqrt{1 + \frac{1}{(C-x^2)^3}}.$$

If $b < 1$ then y is defined for all x while, if $b > 1$, the solution y is defined only for $|x| < \sqrt{3C}$.

2. This is a Bernoulli equation. To solve it, divide both sides by y^3 and make the change of variables $u = 1/y^2$. This gives

$$u' = -2u - 2e^{2x}$$

after multiplication by -2. We now have a linear equation whose general solution is

$$u = -e^{2x}/2 + Ce^{-2x}.$$

This gives a 1-paramenter family of solutions

$$y = \frac{\pm 1}{\sqrt{Ce^{-2x} - e^{2x}/2}} = \frac{\pm e^x}{\sqrt{C - e^{4x}/2}}$$

of the original DE. Given (x_0, y_0) with $y_0 \neq 0$ there is a unique value of C such that the solution satisfies $y(x_0) = y_0$. It is not the general solution as it omits the solution $y = 0$. Thus the general solution is comprised of the functions

$$y = 0, \quad y = \frac{\pm e^x}{\sqrt{C - e^{4x}/2}}.$$

3. This is a homogeneous equation. Setting $u = y/x$, we get

$$xu' + u = 1/u + u.$$

This gives $xu' = 1/u$, a separable equation from which we get $uu' = 1/x$. Integrating, we get

$$u^2/2 = \ln|x| + C_1$$

and hence $y^2 = x^2 \ln(x^2) + Cx^2$ with $C = 2C_1$. For $y(1) = -4$ we must have $C = 16$ and

$$y = -x\sqrt{\ln(x^2) + 16}, \quad x > 0.$$

4. This is a linear equation which is also exact. The general solution is $F(x, y) = C$ where

$$\frac{\partial F}{\partial x} = ye^x - 1, \quad \frac{\partial F}{\partial y} = e^y - 1.$$

Integrating the first equation partially with respect to x we get

$$F(x, y) = ye^x + x + \phi(y)$$

from which $\frac{\partial F}{\partial y} = e^x + \phi'(y) = e^y - 1$ which gives $\phi(y) = -y$ (up to a constant) and hence

$$F(x, y) = ye^x + x - y = C.$$

For $y(1) = 1$ we must have $C = e$ and so the solution is

$$y = \frac{e - x}{e^x - 1}, \quad (x > 0).$$