McGill University
Math 325A: Differential Equations
Notes for Lecture 9
Text: Ch. 13
The Fundamental Existence and Uniqueness Theorem
For n-th order Differential Equations

In this lecture we will state and sketch the proof of the fundamental existence and uniqueness theorem for the n-th order DE

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

The starting point is to convert this DE into a system of first order DE'. Let $y_1 = y, y_2 = y', \dots y^{(n-1)} = y_n$. Then the above DE is equivalent to the system

$$\begin{aligned}\frac{dy_1}{dx} &= y_2\\ \frac{dy_2}{dx} &= y_3\\ &\vdots\\ \frac{dy_n}{dx} &= f(x, y_1, y_2, \dots, y_n).\end{aligned}$$

More generally let us consider the system

$$\begin{aligned}\frac{dy_1}{dx} &= f_1(x, y1, y_2, \dots, y_n)\\ \frac{dy_2}{dx} &= f_2(x, y_1, y_2, \dots, y_n)\\ &\vdots\\ \frac{dy_n}{dx} &= f_n(x, y_1, y_2, \dots, y_n).\end{aligned}$$

If we let $Y = (y_1, y_2, \dots, y_n), F(x, Y) = (f_1(x, Y), f_2(x, Y), \dots, f_n(x, Y))$ and $\frac{dY}{dx} = (\frac{dy_1}{dx}, \frac{dy_2}{dx}, \dots, \frac{dy_n}{dx})$ the system becomes

$$\frac{dY}{dx} = F(x, Y).$$

Theorem. If $f_i(x, y_1, \dots, y_n)$ and $\frac{\partial f_i}{\partial y_j}$ are continuous on the $n+1$-dimensional box

$$R : |x - x_0| < a,\ \ |y_i - c_i| < b, (1 \le i \le n)$$

for $1 \le i, j \le n$ with $|fi(x, y)| \le M$ and

$$|\frac{\partial f_i}{\partial y_1}| + |\frac{\partial f_i}{\partial y_2}| + \dots |\frac{\partial f_i}{\partial y_n}| < L$$

on R for all i, the initial value problem

$$\frac{dY}{dx} = F(x, Y), \quad Y(x_0) = (c_1, c_2, \dots, c_n)$$

has a unique solution on the interval $|x - x_0| \leq h = \min(a, b/M)$.

The proof is exactly the same as for the proof for $n = 1$ if we use the following Lemma in place of the mean value theorem.

Lemma. If $f(x_1, x_2, \ldots x_n)$ and its partial derivatives are continuous on an n-dimensional box R, then for any $a, b \in R$ we have

$$|f(a) - f(b)| \leq (|\frac{\partial f}{\partial x_1}(c)| + \cdots + |\frac{\partial f}{\partial x_b}(c)||a - b|$$

where c is a point on the line between a and b and $|(x_1, \ldots, x_n)| = \max(|x_1|, \ldots, |x_n|)$.

The lemma is proved by applying the mean value theorem to the function $G(t) = f(ta + (1-t)b)$. This gives

$$G(1) - G(0) = G'(c)$$

for some c between 0 and 1. The lemma follows from the fact that

$$G'(x) = \frac{\partial f}{\partial x_1}(a_1 - b_1) + \cdots + \frac{\partial f}{\partial x_1}(a_n - b_n).$$

The Picard iterations $Y_k(x)$ defined by

$$Y_0(x) = Y_0 = (c_1, \ldots, c_n), \quad Y_{k+1}(x) = Y_0 + \int_{x_0}^{x} F(t, Y_k(t))dt,$$

converge to the unique solution Y and

$$|Y(x) - Y_k(x)| \leq (M/L)e^{hL}h^{k+1}/(k+1)!.$$

If $f_1(x, y_1, \ldots, y_)$, $\frac{\partial f_i}{\partial y_j}$ are continuous in the strip $|x - x_0| \leq a$ and there is an L such that

$$|f(x, Y) - f(Z)| \leq L|Y - Z|$$

then h can be taken to be a and $M = max|f(x, Y_0)|$. This happens in the important special case

$$f_i(x, y_1, \ldots, y_n) = a_{i1}(x)y_1 + \cdots + a_{in}(x)y_n + b_i(x).$$

As a corollary of the above theorem we get the following fundamental theorem for n-th order DE's.

Theorem. If $f(x, y_1, \ldots, y_n)$ and $\frac{\partial f}{\partial f_j}$ are continuous on the box

$$R : |x - x_0| \leq a, \ |y_i - c_i| \leq b \ (1 \leq i \leq n)$$

and $|f(x, y_1, \ldots, y_n)| \leq M$ on R, then the initial value problem

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad y^{i-1}(x_0) = c_i \ (1 \leq 1 \leq n)$$

has a unique solution on the interval $|x - x_0| \leq h = \max(a, b/M)$.

Another important application is to the n-th order linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x).$$

In this case $f_1 = y_2,\ f_2 = y_3,\ f_n = p_1(x)y_1 + \cdots p_n(x)y_n + q(x)$ where $p_i(x) = a_{n-i}(x)/a_0(x)$, $q(x) = -b(x)/a_0(x)$.

Theorem. If $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous on an interval I and $a_0(x) \neq 0$ on I then, for any $x_0 \in I$, that is not an endpoint of I, and any scalars $c_1, c_2, \ldots, c_n$, the initial value problem

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x), \quad y^{i-1}(x_0) = c_i \ (1 \leq 1 \leq n)$$

has a unique solution on the interval I.