McGill University
Math 325A: Differential Equations
Notes for Lecture 8

Text: Section 3.6

Euler's Method. In this section we discuss methods for obtaining a numerical solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

at equally spaced points $x_0, x_1, x_2, \ldots, x_N = p, \ldots$ where $x_{n+1} - x_n = h > 0$ is called the step size. In general, the smaller the value of the better the approximations will be but the number of steps required will be larger. We begin by integrating $y' = f(x, y)$ between x_n and x_{n+1}. If $y(x) = \phi(x)$, this gives

$$\phi(x_{n+1}) = \phi(x_n) + \int_{x_n}^{x_{n+1}} f(t, \phi(t))dt.$$

As a first estimate of the integrand we use the value of $f(t, \phi(t))$ at the lower limit x_n, namely $f(x_n, \phi(x_n))$. Now, assuming that we have already found an estimate y_n for $\phi(x_n)$, we get the estimate

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $\phi(x_{n+1})$. It can be shown that

$$|y_n - \phi(x_n)| \le Ch,$$

where C is a constant which depends on p.

The Euler method can be improved if we use the trapezoidal rule for estimating the above integral. Namely,

$$\int_a^b F(x)dx = \frac{1}{2}(F(a) + F(b))(b - a).$$

This leads to the estimate

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y_{n+1})).$$

If we now use the Euler approximation y_{n+1} to compute $f(x_{n+1}, y_{n+1})$, we get

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)).$$

This is known as the improved Euler method. It can be shown that

$$|y_n - \phi(x_n)| \le Ch^2.$$

In general, if y_n is an approximation for $\phi(x_n)$ such that

$$|y_n - \phi(x_n)| \le Ch^p,$$

we say that the approximation is of order p. Thus the Euler method is first order and the improved Euler is second order.

On can obtain higher order approximations by using better approximations for $F(t) = f(t, \phi(t))$ on the interval $[x_n, x_{n+1}]$. For example, the Taylor series approximation

$$F(t) = F(x_n) + F'(x_n)(t - x_n) + F''(x_n)(t - x_n)^2/2 + \cdots + F^{(p-1)}(x_n)(t - x_n)^{p-1}/(p - 1)!$$

yields the approximation

$$y_{n+1} = y_n + hf_1(x_x, y_n) + \frac{h^2}{2} f_2(x_n, y_n) + \cdots + \frac{h^p}{p!} f_{p-1}(x_n, y_n),$$

where

$$f_k(x_n, y_n) = F^{(k-1)}(x_n) = (\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y})^{(k-1)} f(x_n, y_n).$$

It can be show that this approximation is of order p. However it is computationally intensive as one has to compute higher derivatives.

In the case $p = 2$ this formula was simplified by Runge and Kutta to give the second order midpoint approximation

$$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)).$$

In the case $p = 4$ they obtained the 4-th order approximation

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned}
k_1 &= hf(x_n, y_n), \\
k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \\
k_3 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}), \\
k_4 &= hf(x_n + h, y_n + k_3).
\end{aligned}$$

Computationally, it is much simpler than the 4-th order Taylor series approximation from which it is derived.