McGill University
Math 325A: Differential Equations
Notes for Lecture 7

Text: pp. 38-40, Ch. 13

Numerical Methods

Most differential equations cannot be solved in terms of known functions. However solutions do exist in general and we have to be able to compute the values of these functions numerically to any desired degree of accuracy and we will give three methods for doing just that, namely

1. Series Solutions,
2. Picard Iteration,
3. Euler Method.

In this lecture we will treat the first two.

Series Solutions. A function $f(x)$ of one variable x is said to be **analytic** at a point $x = x_0$ if it has a convergent power series expansion

$$f(x) = \sum_0^\infty a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots$$

in some interval $|x - x_0| < h$, $h > 0$. In this case, one can show that $a_n = f^{(n)}/n!$. Such a power series is also called a Taylor series. If $x_0 = 0$, it is also called a MacLaurin Series.

Similarly, a function $f(x, y)$ of two variables is called analytic at the point (x_0, y_0) if it has a convergent power series expansion

$$f(x, y) = \sum_{i,j=0}^\infty a_{ij}(x - x_0)^i(x - x_0)^j = a_{00} + a_{10}(x - x_0) + a_{01}(y - y_0) + \cdots$$

on some rectangle $|x - x_0| < h$, $|y - y_0| < k$, $h, k > 0$. We have

$$a_{ij} = \frac{\frac{\partial f^{i+j}}{\partial x^i \partial y^j}(x_0, y_0)}{i!j!}.$$

If $f(x, y)$, $g(x, y)$ are analytic at (x_0, y_0) and a, b are scalars then $af + bg$, fg are also analytic at (x_0, y_0) as well as f/g provided that $g(x_0, y_0) \neq 0$.

If $f(x, y)$ is analytic at (x_0, y_0) the DE $y' = f(x, y)$ has a power series solution $y(x) = \sum a_n(x - x_0)^n$ with $a_n = y^{(n)}(x_0)$. The formula

$$y^{(n)} = (\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y})^{n-1} f(x, y)$$

allows us to find any of the values $y^{(n)}$ but one cannot find a general formula in most cases. Without this information we cannot determine how well the partial sums of the series expansion approximate the solution $y(x)$ at a point x.

Example. Consider the initial value problem $y' = x + y^2$, $y(0) = 1$. Since $f(x, y)$ is analytic at $(0, 1)$ we look for a solution of the form $y = \sum a_n x^n$. Differentiating $y' = x + y^2$ successively with respect to x, we get $y'' = 1 + 2yy'$, $y''' = 2(y')^2 + 2yy''$. Hence,

$$y(0) = 1,\ y'(0) = 1,\ y''(0) = 3,\ y'''(0) = 8$$

which gives $a_0 = y(0) = 1,\ a_1 = y'(0) = 1,\ a_2 = y''(0)/2 = 3/2,\ a_3 = y'''(0)/6 = 4/3$. Hence,

$$y(x) = 1 + x + 3x^2/2 + 4x^3/3 + \text{ higher terms.}$$

We leave as an exercise for the reader to show that the next term in the series is $17x^4/12$ and not $5x^4/4$ as one might have guessed. We don't have a formula for the n-th term and so we cannot say how good an approximation the first four terms of the above series give.

Picard Iteration. We assume that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on the rectangle

$$R:\ |x - x_0| \le a,\ |y - y_0| \le b$$

Then $|f(x, y)| \le M$, $|\frac{\partial f}{\partial y}(x, y)| \le L$ on R. The initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ is equivalent to the integral equation

$$y = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Let the righthand side of the above equation be denoted by $T(y)$. Then our problem is to find a solution to $y = T(y)$ which is a fixed point problem. To solve this problem we take as an initial approximation to y the constant function $y_0(x) = y_0$ and consider the iterations $y_n = T^n(y_0)$. The function y_n is called the n-th Picard iteration of y_0. For example, for the initial value problem $y' = x + y^2$, $y(0) = 1$ we have

$$y_1(x) = 1 + \int_0^x (t + 1)dt = 1 + x + x^2/2$$

$$y_2(x) = 1 + \int_0^x (t + (1 + t + t^2/2)^2)dt = 1 + x + 3x^2/2 + 2x^3/3 + x^4/4 + x^5/20.$$

Contrary to the power series approximations we can determine just how good the Picard iterations approximate y. In fact, we will see that the Picard iterations converge to a solution of our initial value problem. More precisely we have the following result:

Theorem The Picard iterations $y_n = T^n(y_0)$ converge to a solution y of $y' = f(x, y)$, $y(x_0) = y_0$ on the interval $|x - x_0| \le h = \min(a, b/M)$. Moreover

$$|y(x) - y_n(x)| \le (M/L)e^{hL}(Lh)^{n+1}/(n + 1)!$$

for $|x - x_0| \le h$ and the solution y is unique on this interval.

Proof. We have

$$|y_1 - y_0| = |\int_{x_0}^{x} f(t, y_0)| \le M|x - x_0|$$

since $|f(x, y)| \le M$ on R. Now $|y_1 - y_0| \le b$ if $|x - x_0| \le h$. So $(x, y_1(x))$ is in R if $|x - x_0| \le h$. Similarly, one can show inductively that $(x, y_n(x))$ is in R if $|x - x_0| \le h$. Using the fact that, by the mean value theorem for derivatives,

$$|f(x, z) - f(x, w| \le L|z - w|$$

for all $(x, w), (x, z)$ in R, we obtain

$$|y_2 - y_1| = |\int_{x_0}^{x} (f(t, y_1) - f(t, y_0)| \leq ML|x - x_0|^2/2,$$

$$|y_3 - y_2| = |\int_{x_0}^{x} (f(t, y_2) - f(t, y_1)| \leq ML^2|x - x_0|^3/6$$

and by induction $|y_n - y_{n-1}| \leq ML^{n-1}|x - x_0|^n/n!$. Since the series $\sum_1^\infty |y_n - y_{n-1}|$ is bounded above term by term by the convergent series $(M/L)\sum_1^\infty (L|x-x_0|)^n/n!$, its n-th partial sum $y_n - y_0$ converges, which gives the convergence of y_n to a function y. Now since

$$y = y_0 + (y_1 - y_0) + \cdots + (y_n - y_{n-1}) + \sum_{i=n+1}^{\infty} (y_i - y_{i-1})$$

we obtain

$$|y - y_n| \leq \sum_{i=n+1}^{\infty} (M/L)(L(|x - x_0|)^i/i! \leq (M/L)\frac{(Lh)^{n+1}}{(n+1)!}e^{hL}.$$

For the uniqueness, suppose $T(z) = z$ with $(x, z(x)$ in R for $|x - x_0| \leq h$. Then

$$y(x) - z(x) = \int_{x_0}^{x} (f(t, y(x)) - f(t, z(x))dt.$$

If $|y(x) - z(x)| \leq A$ for $x - x_0| \leq h$ we then obtain as above

$$|y(x) - z(x)| \leq AL|x - x_0|.$$

Now using this estimate, repeat the above to get

$$|y(x) - z(x)| \leq AL^2|x - x_0|^2/2.$$

Using induction we get that

$$|y(x) - z(x)| \leq AL^n|x - x_0|^n/n!$$

which converges to zero for all x. Hence $y = z$. **QED**

The key ingredient in the proof is the **Lipschitz Condition**

$$|f(x, y) - f(x, z)| \leq L|y - z|.$$

If $f(x, y)$ is continuous for $|x - x_0| \leq a$ and all y and satisfies the above Lipschitz condition in this strip the above proof gives the existence and uniqueness of the solution to the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ on the interval $|x - x_0| \leq a$.