McGill University
Math 325A: Differential Equations
Notes for Lecture 6

Text: Sections 3.1,3.2,3.4

We now give a few applications of differential equations.

Falling Bodies with Air Resistance. Let x be the height at time t of a body of mass m falling under the influence of gravity. If g is the force of gravity and $b\frac{dx}{dt}$ is the force on the body due to air resistance, Newton's Second Law of Motion gives the DE

$$m\frac{dv}{dt} = mg - bv$$

where $v = \frac{dx}{dt}$. This DE has the general solution

$$v(t) = \frac{mg}{b} + Be^{-bt}.$$

The limit of $v(t)$ as $t \to \infty$ is mg/b, the terminal velocity of the falling body. Integrating once more, we get

$$x(t) = A + \frac{mgt}{b} - \frac{B}{b}e^{-bt}.$$

Mixing Problems Suppose that a tank is being filled with brine at the rate of a units of volume per second and at the same time b units of volume per second are pumped out. If the concentration of the brine coming in is c units of weight per unit of volume. If at time $t = t_0$ the volume of brine in the tank is V_0 and contains x_0 units of weight of salt, what is the quantity of salt in the tank at any time t, assuming that the tank is well mixed?

If x is the quantity of salt at any time t, we have ac units of weight of salt coming in per second and

$$\frac{bx}{V_0 + (a-b)(t-t_0)}$$

units of weight of salt going out. Hence

$$\frac{dx}{dt} = ac - \frac{bx}{V_0 + (a-b)(t-t_0)},$$

a linear equation. If $a = b$ it has the solution

$$x(t) = cV_0 + (x_0 - cV_0)e^{-a(t-t_0)/V_0}.$$

As a numerical example, suppose $a = b = 1$ liter/min, $c = 1$ grams/liter, $V_0 = 1000$ liters, $x_0 = 0$ and $t_0 = 0$. Then

$$x(t) = 1000(1 - e^{-.001t})$$

is the quantity of salt in the tank at any time t. Suppose that after 100 minutes the tank springs a leak letting out an additional liter of brine per minute. To find out how much salt is in the tank 12 hours after the leak begins we use the DE

$$\frac{dx}{dt} = 1 - \frac{2x}{1000 - (t - 100)} = 1 - \frac{2}{1100 - t}x.$$

This equation has the general solution

$$x(t) = (1100 - t)^{-1} + C(1100 - t)^2.$$

Using $x(100) = 1000(1 - e^{-.1}) = 95.16$, we find $C = -9.048 \times 10^{-4}$ and $x(820) = 177.1$. When $t = 1100$ the tank is empty and the differential equation is no a valid description of the physical process. The concentration at time $100 < t < 1100$ is

$$\frac{x(t)}{1100 - t} = 1 + C(1100 - t)$$

which converges to 1 as t tends to 1100.

Heating and Cooling Problems Newton's Law of Cooling states that the rate of change of the temperature of a cooling body is proportional to the difference between its temperature T and the temperature of its surrounding medium. Assuming the surroundings maintain a constant temperature T_s, we obtain the differential equation

$$\frac{dT}{dt} = -k(T - T_s),$$

where k is a constant. This is a linear DE with solution

$$T = T_s + Ce^{-kt}.$$

If $T(0) = T_0$ then $C = T_0 - T_s$ and

$$T = T_s + (T_0 - T_s)e^{-kt}.$$

As an example consider the problem of determining the time of death of a healthy person who died in his home some time before noon when his body was 70 degrees. If his body cooled another 5 degrees in 2 hours when did he die, assuming that the room was a constant 60 degrees. Taking noon as $t = 0$ we have $T_0 = 70$. Since $T_s = 60$, we get $65 - 60 = 10e^{-2k}$ from which $k = \ln(2)/2$. To determine the time of death we use the equation $98.6 - 60 = 10e^{-kt}$ which gives $t = -\ln(3.86)/k = -2\ln(3.86)/\ln(2) = -3.90$. Hence the time of death was $8:06$ AM.

Radioactive Decay A radioactive substance decays at a rate proportional to the amount of substance present. If x is the amount at time t we have

$$\frac{dx}{dt} = -kx,$$

where k is a constant. The solution of the DE is $x = x(0)e^{-kt}$. If c is the half-life of the substance we have by definition

$$x(0)/2 = x(0)e^{-kc}$$

which gives $k = \ln(2)/c$.