McGill University
Math 325A: Differential Equations
Notes for Lecture 5

Text: Sections 2.5,2.6

Change of Variables. Sometimes is is possible by means of a change of variable to transform a DE into one of the known types. For example, homogeneous equations can be transformed into separable equations and Bernoulli equations can be transformed into linear equations. Another example is a DE of the form

$$\frac{dy}{dx} = f(ax + by), b \neq 0$$

Here, if we make the substitution $u = ax + by$ the differential equation becomes

$$\frac{du}{dx} = bf(u) + a$$

which is separable. For example the DE $y' = 1 + \sqrt{y - x}$ becomes $u' = \sqrt{u}$ after the change of variable $u = y - x$.

Another example is the differential equation

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

where $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ are distinct lines meeting in the point (x_0, y_0). The above DE can be written in the form

$$\frac{dy}{dx} = \frac{a_1(x - x_0) + b_1(y - y_0)}{a_2(x - x_0) + b_2(y - y_0)}$$

which yields the DE

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

after the change of variables $X = x - x_0$, $Y = y - y_0$.

As a final example, we consider the Ricatti equation

$$\frac{dy}{dx} = p(x)y + q(x)y^2 + r(x).$$

Suppose that $u = u(x)$ is a solution of this DE and make the change of variables $y = u + 1/v$. Then $y' = u' - v'/v^2$ and the DE becomes

$$\begin{aligned} u' - v'/v^2 &= p(x)(u + 1/v) + q(x)(u^2 + 2u/v + 1/v^2) + r(x) \\ &= p(x)u + q(x)u^2 + r(x) + (p(x) + 2uq(x))/v + q(x)/v^2 \end{aligned}$$

from which we get $v' + (p(x) + 2uq(x))v = -q(x)$, a linear equation. For example, $y' = 1 + x^2 - y^2$ has the solution $y = x$ and the change of variable $y = x + 1/v$ transforms the equation into $v' + 2xv = 1$.

An important application of first order DE's is to the computation of the orthogonal trajectories of a family of curves $f(x, y, C) = 0$. An orthogonal trajectory of this family is a curve that, at each point of intersection with a member of the given family, intersects that member orthogonally. If we differentiate $f(x, y, C) = 0$ implicitly with respect to x we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0$$

from which we get

$$y' = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Now assume that you can solve for C in the equation $f(x, y, C) = 0$ and substitute in the above formula for y'. This yields the slope of the tangent line at the point (x, y) of a curve of the given family passing through (x, y). Therefore, the slopes of the orthogonal trajectories must satisfy

$$y' = \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$$

which is the DE for the orthogonal trajectories.

For example, let us find the orthogonal trajectories of the family $x^2 + y^2 = Cx$, the family of circles with center on the x-axis and passing through the origin. Here

$$2x + 2yy' = C = \frac{x^2 + y^2}{x}$$

from which $y' = y^2 - x^2)/2xy$. The differential equation for the orthogonal trajectories is then $y' = 2xy/(x^2 - y^2)$ from which

$$2xy + (y^2 - x^2)y' = 0.$$

If we let $M = 2xy$, $N = y^2 - x^2$ we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4x}{2xy} = \frac{2}{y}$$

so that we have an integrating factor μ which is a function of y. We have $\mu' = -2\mu/y$ from which $\mu = 1/y^2$. Multiplying the DE for the orthogonal trajectories by $1/y^2$ we get

$$\frac{2x}{y} + (1 - \frac{x^2}{y^2})y' = 0.$$

Solving $\frac{\partial F}{\partial x} = 2x/y$, $\frac{\partial F}{\partial y} = 1 - x^2/y^2$ for F yields $F(x, y) = x^2/y + y$ from which the orthogonal trajectories are $x^2/y + y = C$, i.e., $x^2 + y^2 = Cy$. This is the family of circles with center on the y-axis and passing through the origin. Note that the line $y = 0$ is also an orthogonal trajectory that was not found by the above procedure. This is due to the fact that the integrating factor was $1/y^2$ which is not defined if $y = 0$ so we had to work in a region which does not cut the x-axis, e.g., $y > 0$ or $y < 0$.