McGill University
Math 325A: Differential Equations
Notes for Lecture 4

Text: Sections 2.4,2.5

Exact Equations. By a region of the xy-plane we mean a connected open subset of the plane. The differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be exact on a region R if there is a function $F(x,y)$ defined on R such that

$$\frac{d}{dx}F(x,y) = M(x,y) + N(x,y)\frac{dy}{dx}.$$

Since

$$\frac{d}{dx}F(x,y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

this is true if $M(x,y) = \frac{\partial F}{\partial x}$, $N(x,y) = \frac{\partial F}{\partial y}$. In this case, if M, N are continuously differentiable on R we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Conversely, it can be shown that this is sufficient for the exactness of the given DE on R providing that R is simply connected, .i.e., has no "holes". Note that $F(x,y)$, if it exists,is determined up to an additive constant and the general solution of the given DE in implicit form is $F(x,y) = C$. The curves $F(x,y) = C$ are called integral curves of the given DE.

Example 1. $2x^2y\frac{dy}{dx} + 2xy^2 + 1 = 0$. Here $M = 2xy^2 + 1$, $N = 2x^2y$ and $R = \mathbb{R}^2$, the whole xy-plane. The equation is exact on $\mathbb{R}^2$ since $\mathbb{R}^2$ is simply connected and

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}.$$

To find F we have to solve the partial differential equations

$$\frac{\partial F}{\partial x} = 2xy^2 + 1, \quad \frac{\partial F}{\partial y} = 2x^2y.$$

If we integrate the first equation with respect to x holding y fixed, we get

$$F(x,y) = x^2y^2 + x + \phi(y).$$

Differentiating this equation with respect to y gives

$$\frac{\partial F}{\partial y} = 2x^2y + \phi'(y) = 2x^2y$$

using the second equation. Hence $\phi'(y) = 0$ and $\phi(y)$ is a constant function. The general solution of our DE in implicit form is $x^2y^2 + x = C$.

Example 2. We have already solved the homogeneous DE

$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

This equation can be written in the form

$$y - x + (x + y)\frac{dy}{dx} = 0$$

which is an exact equation. In this case, the general solution in implicit form is $x(y-x)+y(x+y) = C$, i.e., $y^2 + 2xy - x^2 = C$, using the following result due to Euler.

Theorem. If $F(x, y)$ is homogeous of degree n then

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF(x, y).$$

Proof. The function F is homogeneous of degree n if $F(tx, ty) = t^n F(x, y)$. Differentiating this with respect to t and setting $t = 1$ yields the result. **QED**

Integrating Factors. If the differential equation $M + Ny' = 0$ is not exact it can sometimes be made exact by multiplying it by a continuously differentiable function $\mu(x, y)$. Such a function is called an integrating factor. An integrating factor μ satisfies the PDE $\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$ which can be written in the form

$$(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}.$$

This equation can be simplified in special cases, two of which we treat next.

(a) μ **is a function of** x **only.** This happens if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$$

is a function $p(x)$ of x only in which case $\mu' = p(x)\mu$.

(b) μ **is a function of** y **only.** This happens if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M}$$

is a function $q(y)$ of y only in which case $\mu' = -q(y)\mu$.

Example 1. $2x^2 + y + (x^2y - x)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2}{x}$$

so that there is an integrating factor μ which is a function of x only which satisfies $\mu' = -2\mu/x$. Hence $\mu = 1/x^2$ is an integrating factor and $2 + y/x^2 + (y - 1/x)y' = 0$ is an exact equation whose general solution is $2x - y/x + y^2/2 = C$ or $2x^2 - y + xy^2/2 = Cx$.

Example 2.$y + (2x - ye^y)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y}$$

so that there is an integrating factor which is a function of y only which satisfies $\mu' = 1/y$. Hence y is an integrating factor and $y^2 + (2xy - y^2 e^y)y' = 0$ is an exact equation with general solution $xy^2 + (-y^2 + 2y - 2)e^y = C$.

A word of caution is in order here. The solutions of the exact DE obtained by multiplying by the integrating factor may have solutions which are not solutions of the original DE. This is due to the fact that μ may be zero and one will have to possibly exclude those solutions where μ vanishes. For example in the second example above, one gets a $y = 0$ of the exact DE which is not a solution of the original DE.