McGill University
Math 325A: Differential Equations
Notes for Lecture 3

Text: Sections 1.1,2.2,2.6

We now give some more examples of separable equations. We begin with the **logistic equation**

$$y' = ay(b-y)$$

where $a, b > 0$ are fixed constants. This equation arises in the study of the growth of certain populations. Since the right-hand side of the equation is zero for $y = 0$ and $y = b$, the given DE has $y = 0$ and $y = b$ as solutions. More generally, if $y' = f(x, y)$ and $f(x, c) = 0$ for all x in some interval I, the constant function $y = c$ on I is a solution of $y' = f(x, y)$ since $y' = 0$ for a constant function y.

To solve the logistic equation, we write it in the form

$$\frac{y'}{y(b-y)} = a.$$

Integrating both sides with respect to x we get

$$\int \frac{y'dx}{y(b-y)} = ax + C$$

which can, since $y'dx = dy$, be written as

$$\int \frac{dy}{b-y} = ax + C.$$

Since, by partial fractions,

$$\frac{1}{y(b-y)} = \frac{1}{b}(\frac{1}{y} + \frac{1}{b-y})$$

we obtain

$$\frac{1}{b}(\ln|y| - \ln|b-y|) = ax + C_1.$$

Multiplying both sides by b and exponentiating both sides to the base e, we get

$$\frac{|y|}{|b-y|} = e^{C_1}e^{abx}$$

from which $y/(b-y) = \pm e^{C_1}e^{abx}$. The constant $C = \pm e^{C_1}$ can be any non-zero scalar. We now have

$$y = (b-y)Ce^{abx} = bCe^{abx} - Cye^{abx}$$

and hence $y(1 + Ce^{abx}) = bCe^{abx}$. This gives

$$y = \frac{bCe^{abx}}{1 + Ce^{abx}}.$$

However, this is not the general solution as the solutions $y = 0$ and $y = b$ are not of the above form for any non-zero constant C. Since the above equation reduces to $y = 0$ if $C = 0$, we do recover one of the omitted solutions if we allow C to be zero. But is

$$y = b, \quad y = \frac{bCe^{abx}}{1 + Ce^{abx}}$$

the general solution? To see that it is, it suffices to notice that for any x_0, y_0 there is exactly one of these solutions for which $y(x_0) = y_0$ and to appeal to the following theorem which guarantees the existence and uniquess of solutions of solutions to the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ under certain conditions.

Fundamental Existence and Uniqueness Theorem. If the function $f(x, y)$ together with its partial derivative with respect to y are continuous on the rectangle

$$R : |x - x_0| \leq a,\ \ |y - y_0| \leq b$$

there is a unique solution to the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

defined on the interval $|x - x_0| < h$ where

$$h = \min(a, b/M), \quad M = \max |f(x, y)|,\ (x, y) \in R.$$

Note that this theorem indicates that a solution may not be defined for all x in the interval $|x - x_0| \leq a$. For example, the function

$$y = \frac{bCe^{abx}}{1 + Ce^{abx}}$$

is solution to $y' = ay(b - y)$ but not defined when $1 + Ce^{abx} = 0$ even though $f(x, y) = ay(b - y$ satisfies the conditions of the theorem for all x, y.

The next example show why the condition on the partial derivative in the above theorem is necessary.

Consider the differential equation $y' = y^{1/3}$. Again $y = 0$ is a solution. Separating variables and integrating, we get

$$\int \frac{dy}{y^{1/3}} = x + C_1$$

which yields $y^{2/3} = 2x/3 + C$ and hence $y = \pm(2x/3 + C)^{3/2}$. Taking $C = 0$, we get the solution $y = (2x/3)^{3/2}, (x \geq 0)$ which along with the solution $y = 0$ satisfies $y(0) = 0$. So the initial value problem $y' = y^{1/3},\ y(0) = 0$ does not have a unique solution. The reason this is so is due to the fact that $\frac{\partial f}{\partial y}(x, y) = 1/3y^{2/3}$ is not continuous when $y = 0$.

Many differential equations become linear or separable after a change of variable. We now give two examples of this.

Bernoulli Equation: $y' = p(x)y + q(x)y^{\alpha}$ $(\alpha \neq 1)$. Note that $y = 0$ is a solution. To solve this equation, divide both sides by y^{α} to get

$$y^{-\alpha}y' = p(x)y^{1-\alpha} + q(x).$$

Setting $u = y^{1-\alpha}$ we have $u' = (1 - \alpha)y^{-\alpha}y'$ so that our differential equation becomes

$$u'/(1 - \alpha) = p(x)u + q(x)$$

which is linear. We know how to solve this for u from which we get solve $u = y^{1-\alpha}$ to get y.

Homogeneous Equation: $y' = F(y/x)$. To solve this we let $u = y/x$ so that $y = xu$ and $y' = u + xu'$. Substituting for y, y' in our DE gives $u + xu' = F(u)$ which is a separable equation. Solving this for u gives y via $y = xu$. Note that $u = a$ is a solution of $xu' = F(u) - u$ whenever $F(a) = a$ and that this gives $y = ax$ as a solution of $y' = f(y/x)$.

Example. $y' = (x - y)/x + y$. This is a homogeneous equation since

$$\frac{x-y}{x+y} = \frac{1-y/x}{1+y/x}.$$

Setting $u = y/x$, our DE becomes

$$xu' + u = \frac{1-u}{1+u}$$

so that

$$xu' = \frac{1-u}{1+u} - u = \frac{1-2u-u^2}{1+u}.$$

Note that the right-hand side is zero if $u = -1 \pm \sqrt{2}$. Separating variables and integrating with respect to x, we get

$$\int \frac{(1+u)du}{1-2u-u^2} = \ln|x| + C_1$$

which in turn gives

$$(-1/2)\ln|1-2u-u^2| = \ln|x| + C_1.$$

Exponentiating, we get

$$\frac{1}{\sqrt{|1-2u-u^2|}} = e^{C_1}|x|.$$

Squaring both sides and taking reciprocals, we get

$$u^2 + 2u - 1 = C/x^2$$

with $C = \pm 1/e^{2C_1}$. This equation can be solved for u using the quadratic formula. If x_0, y_0 are given with $x_0 \neq 0$ and $u_0 = y_0/x_0 \neq -1$ there is, by the fundamental, existence and uniqueness theorem,a unique solution with $u(x_0) = y_0$. For example, if $x_0 = 1, y_0 = 2$, we have $C = 7$ and hence

$$u^2 + 2u - 1 = 7/x^2$$

Solving for u, we get

$$u = -1 + \sqrt{2 + 7/x^2}$$

where the positive sign in the quadratic formula was chosen to make $u = 2, x = 1$ a solution. Hence

$$y = -x + x\sqrt{2 + 7/x^2} = -x + \sqrt{2x^2 + 7}$$

is the solution to the initial value problem

$$y' = \frac{x-y}{x+y}, \quad y(1) = 2$$

for $x > 0$ and one can easily check that it is a solution for all x. Moreover, using the fundamental uniqueness, it can be shown that it is the only solution defined for all x.