McGill University
Math 325A: Differential Equations
Notes for Lecture 22
Text: Ch. 8

Bessel Functions

In this lecture we study an important class of functions which are defined by the differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where $\nu \geq 0$ is a fixed parameter. This DE is known **Bessel's equation of order** ν; do not confuse ν with the order of the DE which is 2. This equation has $x = 0$ as its only singular point. Moreover, this singular point is a regular singular point since

$$xp(x) = 1, \quad x^2q(x) = x^2 - \nu^2.$$

Bessel's equation can also be written

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2}) = 0$$

which for x large is approximately the DE $y'' + y = 0$ so that we can expect the solutions to oscillate for x large. The indicial equation is $r(r-1) + r - \nu^2 = r - \nu^2$ whose roots are $r_1 = \nu$ and $r_2 = -\nu$. The recursion equations are

$$((1+r)^2 - \nu^2)a_1 = 0, \quad ((n+r)^2 - \nu^2)a_n = -a_{n-2}, \quad \text{for } n \geq 2.$$

The general solution of these equations is $a_{2n+1} = 0$ for $n \geq 0$ and

$$a_{2n}(r) = \frac{(-1)^n a_0}{(r+2-\nu)(r+4-\nu)\cdots(r+2n-\nu)(r+2+\nu)(r+4+\nu)\cdots(r+2n+\nu)}.$$

If ν is not an integer, we obtain two linearly independent solutions of Bessel's equation $J_\nu(x)$, $J_{-\nu}(x)$ by taking $r = \pm\nu$, $a_0 = 1/2^\nu\Gamma(\nu+1)$. Since, in this case,

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n}n!(r+1)(r+2)\cdots(r+n)},$$

we have for $r = \pm\nu$

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(r+n+1)}(\frac{x}{2})^{2n+r}.$$

These functions are called **Bessel functions of first kind of order** ν**.** For $\nu = 0$ we have

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2}x^{2n}$$

and for $\nu = 1$

$$J_1(x) = \frac{x}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}n!(n+1)!}x^{2n}.$$

Recall that the Gamma function $\Gamma(x)$ is defined for $x \geq -1$ by

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt.$$

For $x \geq 0$ we have $\Gamma(x+1) = x\Gamma(x)$, so that $\Gamma(n+1) = n!$ for n an integer ≥ 0. We have

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t} t^{-1/2} dt = 2\int_0^\infty e^{-x^2} dt = \sqrt{\pi}.$$

The Gamma function can be extended uniquely for all x except for $x = 0, -1, -2, \dots, -n, \dots$ to a function which satisfies the identity $\Gamma(x) = \Gamma(x)/x$. This is true even if x is taken to be complex. The resulting function is analytic except at zero and the negative integers where it has a simple pole.

As an exercise the reader can show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x), \quad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}}\sin(x).$$

For $\nu = -m$ with m an integer ≥ 0 one has to proceed differently to get a second solution. For $\nu = 0$ the indicial equation has a repeated root and one has a second solution of the form

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^\infty a'_{2n}(0)x^{2n}$$

where

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2\cdots(r+2n)^2}.$$

It follows that

$$\frac{a'_{2n}(r)}{a_{2n}} = -2(\frac{1}{r+2} + \frac{1}{r+4} + \cdots \frac{1}{r+2n}$$

so that

$$a'_{2n}(0) = (1 + \frac{1}{2} + \cdots + \frac{1}{n})a_{2n}(0) = h_n a_{2n}(0).$$

Hence

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^\infty \frac{(-1)^n h_n}{2^{2n}(n!)^2}x^{2n}.$$

Instead of y_2, the second solution is usually taken to be a certain linear combination of y_2 and J_0. For example, the function

$$Y_0(x) = \frac{2}{\pi}(y_2(x) + (\gamma - \ln(2))J_0(x)),$$

where $\gamma = \lim_{n\to\infty}(h_n - \ln(n) \approx 0.5772$, is known as the **Weber function of order** 0. The constant γ is known as Euler's constant; it is not known whether γ is rational or not.

If $\nu = -m$, with $m > 0$, the the roots of the indicial equation differ by an integer and one has a solution of the form

$$y_2 = aJ_m(x)\ln(x) + \sum_{n=0}^\infty b'_{2n}(-m)x^{2n+m}$$

where $b_{2n}(r) = (r+m)a_{2n}(r)$ and $a = b_{2m}(-m)$. In the case $m = 1$ we have

$$b_{2n}(r) = \frac{(-1)^n a_0}{(r+3)(r+5)\cdots(r+2n-1)(r+3)(r+5)\cdots(r+2n+1)},$$

$$b'_{2n}(r) = -(\frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n-1} + \frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n+1})b_{2n}(r),$$

$$\begin{aligned} b'_{2n}(-1) &= \frac{-1}{2}(h_n + h_{n-1})b_{2n}(-1), \\ &= \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n+1}(n-1)!n!} \end{aligned}$$

so that

$$y_2 = \frac{-1}{2}J_1(x)\ln(x) + \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n+1}(n-1)!n!}x^{2n}$$

where, by convention, $h_{-1} = h_0 = 0$, $(-1)! = 1$. The **Weber function of order 1** is defined to be

$$Y_1(x) = \frac{4}{\pi}(-y_2(x) + (\gamma - \ln(2)J_1(x)).$$

The case $m > 1$ is slightly more complicated and will not be treated here.

The second solutions $y_2(x)$ of Bessel's equation of order $m \geq 0$ are unbounded as $x \to 0$. It follows that any solution of Bessel's equation of order $m \geq 0$ which is bounded as $x \to 0$ is a scalar multiple of J_m. The solutions which are unbounded as $x \to 0$ are called **Bessel functions of the second kind**. The Weber functions are Bessel functions of the second kind.