McGill University
Math 325A: Differential Equations
Notes for Lecture 20
Text: Ch. 8

Series Solution of Linear Differential Equations.

A function $f(x)$ of one variable x is said to be **analytic** at a point $x = x_0$ if it has a convergent power series expansion

$$f(x) = \sum_0^\infty a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots$$

for $|x - x_0| < R$, $R > 0$. Otherwise, $f(x)$ is said to have a **singularity** at $x = x_0$. The largest such R (possibly $+\infty$) is called the **radius of convergence** of the power series. The series converges for every x with $|x - x_0| < R$ and diverges for every x with $|x - x_0| > R$. There is a formula for R given by

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}$$

if the latter limit exists. The same is true if $x,\ x_0,\ a_i$ are complex. For example,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

for $|x| < 1$. The radius of convergence of the series is 1. It is also equal to the distance from 0 to the nearest singularity $x = i$ of $1/(x^2+1)$ in the complex plane.

Power series can be integrated and differentiated within the interval (disk) of convergence. More precisely, for $|x - x_0| < R$ we have

$$f'(x) = \sum_{n=1}^\infty n a_n x^{n-1} = \sum_{n=0}^\infty (n+1) a_{n+1} x^n,$$

$$\int_0^x \sum_{n=0}^\infty a_n t^n = \sum_{n=0}^\infty \frac{a_n}{n+1} x^{n+1} = \sum_{n=1}^\infty \frac{a_{n-1}}{n} x^n$$

and the resulting power series have R as radius of convergence. If $f(x)$, $g(x)$ are analytic at $x = x_0$ then so is $f(x)g(x)$ and $af + bg$ for any scalars a, b with radii of convergence at least that of the smaller of the radii of convergence the series for $f(x), g(x)$. If $f(x)$ is analytic at $x = x_0$ and $f(x_0) \neq 0$ then $1/f(x_0)$ is analytic at $x = x_0$ with radius of convergence equal to the distance from x_0 to the nearest zero of $f(x)$ in the complex plane.

The following theorem shows that linear DE's with analytic coefficients at x_0 have analytic solutions at x_0 with radius of convergence as big as the smallest of the radii of convergence of the coefficient functions.

Theorem. If $p_1(x), p_2(x), \ldots, p_n(x), q(x)$ are analytic at $x = x_0$, the solutions of the DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = q(x)$$

are analytic with radius of convergence $\geq$ the smallest of the radii of convergence of the coefficient functions $p_1(x), p_2(x), \ldots, p_n(x), q(x)$.

The proof of this result follows from the proof of fundamental existence and uniqueness theorem for linear DE's using elementary properties of analytic functions and the fact that uniform limits of analytic functions are analytic.

Example 1. The coefficients of the DE $y'' + y = 0$ are analytic everywhere, in particular at $x = 0$. Any solution $y = y(x)$ has therefore a series representation

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with infinite radius of convergence. We have

$$y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n, \quad y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n.$$

Therefore, we have

$$y'' + y = \sum_{n=0}^{\infty} ((n+1)(n+2)a_{n+2} + a_n)x^n = 0$$

for all x. It follows that $(n+1)(n+2)a_{n+2} + a_n = 0$ for $n \geq 0$. Thus

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \quad \text{for } n \geq 0$$

from which we obtain

$$a_2 = -\frac{a_0}{1 \cdot 2}, \; a_3 = -\frac{a_1}{2 \cdot 3}, \; a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4}, \; a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}.$$

By induction one obtains

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}, \quad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$$

and hence that

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = a_0 \cos(x) + a_1 \sin(x).$$

Example 2. The simplest non-constant DE is $y'' - xy = 0$ which is known as Airy's equation. Its coefficients are analytic everywhere and so the solutions have a series representation

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with infinite radius of convergence. We have

$$\begin{aligned} y'' - xy &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1}, \\ &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1} x^n, \\ &= 2a_2 + \sum_{n=1}^{\infty} ((n+1)(n+2)a_{n+2} + a_{n-1})x^n = 0 \end{aligned}$$

from which we get $a_2 = 0$, $(n+1)(n+2)a_{n+2} + a_{n-1} = 0$ for $n \geq 1$. Since $a_2 = 0$ and

$$a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}, \quad \text{for } n \geq 1$$

we have

$$a_3 = -\frac{a_0}{2 \cdot 3},\ a_4 = -\frac{a_1}{3 \cdot 4}, a_5 = 0, a_6 = -\frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, a_7 = -\frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}.$$

By induction we get $a_{3n+2} = 0$ for $n \geq 0$ and

$$a_{3n} = (-1)^n \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) \cdot 3n}, \quad a_{3n+1} = (-1)^n \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n) \cdot (3n+1)}.$$

Hence $y = a_0 y_1 + a_1 y_2$ with

$$y_1 = 1 - \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} - \cdots + (-1)^n \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) \cdot 3n} + \cdots,$$

$$y_2 = x - \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} - \cdots (-1)^n \frac{x^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n) \cdot (3n+1)} + \cdots.$$

For negative x the solutions of the DE $y'' - xy = 0$ behave like the solutions to a mass-spring system with variable spring constant. The solutions oscillate for $x < 0$ with increasing frequency as $|x| \to \infty$. For $x > 0$ the solutions are monotone. For example, y_1, y_2 are increasing functions of x for $x \geq 0$.