McGill University
Math 325A: Differential Equations
Notes for Lecture 2

Text: Sections 2.1, 2.2

In this lecture we will treat linear and separable first order ODE's.

Linear Equations. The general first order ODE has the form $F(x, y, y') = 0$ where $y = \phi(x)$. If it is linear it can be written in the form

$$a_0(x)y' + a_1(x)y = b(x)$$

where $a_0(x),\ a_(x),\ b(x)$ are continuous functions of x on some interval I. To bring it to normal form $y' = f(x, y)$ we have to divide both sides of the equation by $a_0(x)$. This is possible only for those x where $a_0(x) \neq 0$. After possibly shrinking I we assume that $a_0(x) \neq 0$ on I. So our equation has the form (standard form)

$$y' + p(x)y = q(x)$$

with $p(x) = a_1(x)/a_0(x)$ and $q(x) = b(x)/a_0(x)$, both continous on I. Solving for y' we get the normal form for a linear first order ODE, namely

$$y' = q(x) - p(x)y.$$

We now introduce the function, which is called an integrating factor,

$$\mu(x) = e^{\int p(x)dx}$$

It has the property $\mu'(x) = p(x)\mu(x)$ and $\mu(x) \neq 0$ for all x. Hence our differential equation is eqivalent (has the same solutions) to the equation

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x).$$

Since the left hand side of this equation is the derivative of $\mu(x)y$, it can be written in the form

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x).$$

Integrating both sides, we get

$$\mu(x)y = \int \mu(x)q(x)d(x) + C$$

with C an arbitrary constant. Solving for y, we get

$$y = \frac{1}{\mu(x)} \int \mu(x)q(x)d(x) + \frac{C}{\mu(x)}$$

as the general solution for the general linear first order ODE

$$y' + p(x)y = q(x).$$

Note that for any pair of scalars a, b with a in I, there is a unique scalar C such that $y(a) = b$. Geometrically, this means that the solution curves $y = \phi(x)$ are a family of non-intersecting curves which fill the region $I \times \mathbb{R}$.

Example 1: $y' + xy = x$. This is a linear first order ODE in standard form with $p(x) = q(x) = x$. The integrating factor is

$$\mu(x) = e^{\int x dx} = e^{x^2/2}.$$

Hence, after multiplying both sides of our differential equation, we get

$$\frac{d}{dx}(e^{x^2/2}y) = xe^{x^2/2}$$

which, after integrating both sides, yields

$$e^{x^2/2}y = \int xe^{x^2/2}dx + C = e^{x^2/2} + C.$$

Hence the general solution is $y = 1 + Ce^{-x^2/2}$. The solution satisfying the initial condition $y(0) = 1$ is $y = 1$ and the solution satisfying $y(0) = a$ is $y = 1 + (a-1)e^{-x^2/2}$.

Example 2: $xy' - 2y = x^3 \sin(x)$,
$(x > 0)$. We bring this linear first order equation to standard form by dividing by x. We get

$$y' + \frac{-2}{x}y = x^2 \sin(x).$$

The integrating factor is

$$\mu(x) = e^{\int -2dx/x} = e^{-2\ln(x)} = 1/x^2.$$

After multiplying our DE in standard form by $1/x^2$ and simplifying, we get

$$\frac{d}{dx}(y/x^2) = \sin(x)$$

from which $y/x^2 = -\cos(x) + C$ and $y = -x^2\cos(x) + Cx^2$. Note that the later are solutions to the DE $xy' - 2y = x^3 \sin(x)$ and that they all satisfy the initial condition $y(0) = 0$. This non-uniqueness is due to the fact that $x = 0$ is a singular point of the DE.

Separable Equations. The first order ODE $y' = f(x, y)$ is said to be separable if $f(x, y)$ can be expressed as a product of a function of x times a function of y. The DE then has the form $y' = g(x)h(y)$ and, dividing both sides by $h(y)$, it becomes

$$\frac{y'}{h(y)} = g(x).$$

Of course this is not valid for those solutions $y = \phi(x)$ at the points where $\phi(x) = 0$. Assuming the continuity of g and h, we can integrate both sides of the equation to get

$$\int \frac{y'}{h(y)}dx = \int g(x)dx + C.$$

This will in general give y implicitly in terms of x and one has to solve this implicit equation for y to get y explicitly as a function of x.

Example 1: $y' - ay = 0$. This is a linear first order DE with general solution $y = Ce^{ax}$ as can easily be seen by the first section. Since $y' = ay$ it is also a separable equation. To solve it using the above method we divide both sides of the equation by y to get

$$\frac{y'}{y} = a.$$

Integrating both sides we get $\ln|y| = ax + C$. Exponentiating both sides with base e we get $|y| = e^{ax+C} = e^C e^{ax}$ so that $y = \pm e^C e^{ax}$. The constant $\pm e^C$ can be any non-zero scalar. These solutions are never zero and the are the only solutions which don't vanish at some point. The method does not find the solution $y = 0$. In this case it can be found by inspection.

Example 2: $y' = \frac{y-1}{x+3}$ $(x > -3)$. By inspection, $y = 1$ is a solution. Dividing both sides of the given DE by $y - 1$ we get

$$\frac{y'}{y-1} = \frac{1}{x+3}.$$

This will be possible for those x where $y(x) \neq 1$. Integrating both sides we get

$$\int \frac{y'}{y-1}dx = \int \frac{dx}{x+3} + C,$$

from which we get $\ln|y-1| = \ln(x+3)+C$. Thus $|y-1| = e^C(x+3)$ from which $y-1 = \pm e^C(x+3)$. If we let $A = \pm e^C$, we get

$$y = 1 + A(x + 3)$$

which a family lines passing through $(-3, 1)$; for any (a, b) with $b \neq 0$ there is only one member of this family which passes through (a, b). Since $y = 1$ was found to be a solution by inspection the general solution is

$$y = 1 + C(x + 3),$$

where C can be any scalar.