McGill University
Math 325A: Differential Equations
Notes for Lecture 18
Text: Ch. 7

In this lecture we will show how to use Laplace transforms in solving differential equations. Consider the initial value problem

$$y'' + y' + y = \sin(t), \quad y(0) = 1,\ y'(0) = -1.$$

If $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s) - 1, \quad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.$$

Hence taking Laplace transforms of the DE, we get

$$(s^2 + s + 1)Y(s) - s = \frac{1}{s^2+1}.$$

Solving for $Y(s)$, we get

$$Y(s) = \frac{s}{s^2+s+1} + \frac{1}{(s^2+s+1)(s^2+1)}.$$

Hence

$$y(t) = \mathcal{L}^{-1}\{\frac{s}{s^2+s+1}\} + \mathcal{L}^{-1}\{\frac{1}{(s^2+s+1)(s^2+1)}\}.$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+1/2)^2+3/4} = \frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}$$

we have

$$\mathcal{L}^{-1}\{\frac{s}{s^2+s+1}\} = e^{-t/2}\cos(\sqrt{3}\ t/2) - \frac{1}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2+1}.$$

Multiplying both sides by $(s^2+1)(s^2+s+1)$ and collecting terms, we find

$$1 = (A+C)s^3 + (B+C+D)s^2 + (A+C+D)s + B + D.$$

Equating coefficients, we get $A + C = 0$, $B + C + D = 0$, $A + C + D = 0$, $B + D = 1$, from which we get $A = B = 1$, $C = -1$, $D = 0$ so that

$$\mathcal{L}^{-1}\{\frac{1}{(s^2+s+1)(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{s}{s^2+s+1}\} + \mathcal{L}^{-1}\{\frac{1}{s^2+s+1}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\}.$$

Since

$$\mathcal{L}^{-1}\{\frac{1}{s^2+s+1}\} = \frac{2}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2), \quad \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = \cos(t)$$

we obtain

$$y(t) = 2e^{-t/2}\cos(\sqrt{3}\,t/2) - \cos(t).$$

As a second example, consider the system

$$\begin{aligned}\frac{dx}{dt} &= -2x + y,\\ \frac{dy}{dt} &= x - 2y\end{aligned}$$

with the initial conditions $x(0) = 1$, $y(0) = 2$. Taking Laplace transforms the system becomes

$$\begin{aligned} sX(s) - 1 &= -2X(s) + Y(s),\\ sY(s) - 2 &= X(s) - 2Y(s),\end{aligned}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$. This linear system of equations for $X(s)$, $Y(s)$ can be

$$\begin{aligned}(s+2)X(s) - Y(s) &= 1,\\ -X(s) + (s+2)Y(s) &= 2.\end{aligned}$$

The determinant of the coefficient matrix is $s^2 + 4s + 3 = (s+1)(s+3)$. Using Cramer's rule we get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

Since

$$\frac{s+4}{(s+1)(s+3)} = \frac{3/2}{s+1} - \frac{1/2}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3/2}{s+1} + \frac{1/2}{s+3},$$

we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

The Laplace transform reduces the solution of differential equations to a partial fractions calculation. If $F(s) = P(s)/Q(s)$ is a ratio of polynomials with the degree of $P(s)$ less than the degree of $Q(s)$ then $F(s)$ can be written as a sum of terms each of which corresponds to an irreducible factor of $Q(s)$. Each factor $Q(s)$ of the form $s - a$ contributes the terms

$$\frac{A_1}{s-a} + \frac{A_1}{(s-a)^2} + \cdots + \frac{A_r}{(s-a)^r}$$

where r is the multiplicity of the factor $s-a$. Each irreducible quadratic factor s^2+as+b contributes the terms

$$\frac{A_1 s + B_1}{s^2+as+b} + \frac{A_2 s + B_2}{(s^2+as+b)^2} + \cdots + \frac{A_r s + B_r}{(s^2+as+b)^r}$$

where r is the degree of multiplicity of the factor $s^2 + as + b$.