McGill University
Math 325A: Differential Equations
Notes for Lecture 13
Text: Ch. 4,6
Variation of Parameters

Variation of parameters is method for producing a particular solution of a special kind for the general linear DE in normal form

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$$

from a fundamental set $y_1, y_2, \ldots, y_n$ of solutions of the associated homogeneous equation. In this method we try for a solution of the form

$$y_P = C_1(x)y_1 + C_2(x)y_2 + \cdots + C_n(x)y_n.$$

Then $y_P' = C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n' + C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n$ and we impose the condition

$$C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n = 0.$$

Then $y_P' = C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n'$ and hence

$$y_P'' = C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'' + C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n'.$$

Again we impose the condition $C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' = 0$ so that

$$y_P'' = C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'.$$

We do this for the first $n-1$ derivatives of y so that for $1 \le k \le n-1$

$$y_P^{(k)} = C_1(x)y_1^{(k)} + C_2(x)y_2^{(k)} + \cdots C_n(x)y_n^{(k)},$$

$$C_1'(x)y_1^{(k)} + C_2'(x)y_2^{(k)} + \cdots + C_n'(x)y_n^{(k)} = 0.$$

Now substituting $y_P, y_P', \ldots, y_P^{(n-1)}$ in $L(y) = b(x)$ we get

$$C_1(x)L(y_1) + C_2(x)L(y_2) + \cdots + C_n(x)L(y_n) + C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x).$$

But $L(y_i) = 0$ for $1 \le k \le n$ so that

$$C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x).$$

We thus obtain the system of n linear equations for $C_1'(x), \ldots, C_n'(x)$

$$\begin{aligned}
C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n &= 0,\\
C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' &= 0,\\
&\vdots\\
C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} &= b(x).
\end{aligned}$$

If we solve this system using Cramer's Rule and integrate, we find

$$C_i(x) = \int_{x_0}^{x} (-1)^{n+i} \frac{W_i b(t)}{W} dt$$

where $W = W(y_1, y_2, \dots, y_n)$ and $W_i = W(y_1 \dots, \hat{y_i}, \dots, y_n)$ where the ˆ means that y_i is omitted. Note that the particular solution y_P found in this way satisfies

$$y_P(x_0) = y'_P(x_0) = \cdots = y_P^{(n-1)} = 0.$$

The point x_0 is any point in the interval of continuity of the $a_i(x)$ and $b(x)$. Note that y_P is a linear function of the function $b(x)$.

Example. Find the general solution of $y'' + y = 1/x$ on $x > 0$.

The general solution of $y'' + y = 0$ is $y = c_1 \cos(x) + c_2 \sin(x)$. Using variation of parameters with $y_1 = \cos(x)$, $y_2 = \sin(x)$, $b(x) = 1/x$ and $x_0 = 1$, we have $W = 1$, $W_1 = \sin(x)$, $W_2 = \cos(x)$ and we obtain the particular solution $y_p = C_1(x)\cos(x) + C_2(x)\sin(x)$ where

$$C_1(x) = -\int_1^x \frac{\sin(t)}{t} dt, \quad C_2(x) = \int_1^x \frac{\cos(t)}{t} dt.$$

The general solution of $y'' + y = 1/x$ on $x > 0$ is therefore

$$y = c_1 \cos(x) + c_2 \sin(x) - (\int_1^x \frac{\sin(t)}{t} dt) \cos(x) + (\int_1^x \frac{\cos(t)}{t} dt) \sin(x).$$

When applicable, the annihilator method is easier as one can see from the DE $y'' + y = e^x$. Here it is immediate that $y_p = e^x/2$ is a particular solution while variation of parameters gives

$$y_p = -(\int_0^x e^t sin(t) dt) \cos(x) + (\int_0^x e^t \cos(t) dt) \sin(x).$$

The integrals can be evaluated using integration by parts:

$$\begin{aligned}\int_0^x e^t \cos(t) dt &= e^x \cos(x) - 1 + \int_0^x e^t \sin(t) dt \\ &= e^x \cos(x) + e^x \sin(x) - 1 - \int_0^x e^t \cos(t) dt\end{aligned}$$

which gives

$$\int_0^x e^t \cos(t) dt = (e^x \cos(x) + e^x \sin(x) - 1)/2$$

$$\int_0^x e^t \sin(t) dt = e^x \sin(x) - \int_0^x e^t \cos(t) dt = (e^x \sin(x) - e^x \cos(x) + 1)/2$$

so that after simplification $y_p = e^x/2 - \cos(x)/2 - \sin(x)/2$.

We now give an application of the theory of second order DE's to the description of the motion of a simple mass-spring mechanical system with a damping device. We assume that the damping force is proportional to the velocity of the mass. If there are no external forces we obtain the differential equation

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

where $x = x(t)$ is the displacement from equilibrium at time t of the mass of $m > 0$ units, $b \geq 0$ is the damping constant and $k > 0$ is the spring constant. In operator form with $D = \frac{d}{dt}$ this DE is, after normalizing,

$$(D^2 + \frac{b}{m}D + \frac{k}{m})(x) = 0.$$

The characteristic polynomial $r^2 + (b/m)r + k/m$ has discriminant

$$\Delta = (b^2 - 4km)/m^2.$$

If $b^2 < 4km$ we have $\Delta < 0$ and the characteristic polynomial factorizes in the form $(r+b/2m)^2+\omega^2$ with

$$\omega = \sqrt{4km - b^2}/2m = \sqrt{\frac{k}{m} - (b/2m)^2}.$$

In this case the characteristic polynomial has complex roots $-b/2m \pm i\omega$ and the general solution of the DE is

$$y = e^{-bt/2m}(A\cos(\omega t) + B\sin(\omega t) = Ce^{-bt/2m}\sin(\omega t + \theta)$$

where $C = \sqrt{A^2 + B^2}$ and $0 \leq \theta \leq 2\pi$ defined by $\cos(\theta) = A/C$, $\sin(\theta) = B/C$. The angle θ is called the **phase**. In this case we see that the mass oscillates with **frequency** $\omega/2\pi$ and decreasing amplitude. If $b = 0$ there is no damping and the mass oscillates with frequency $\omega/2\pi$ and constant amplitude; such motion is called **simple harmonic**.

If $b^2 \geq 4km$ we have $\Delta \geq 0$ and so the characteristic polynomial has real roots

$$r_1 = -b/2m + \sqrt{b^2 - 4km}/2m, \quad r_2 = -b/2m - \sqrt{b^2 - 4km}/2m$$

which are both negative. If $r_1 = r_2 = r$ the general solution of our DE is

$$y = Ae^{rt} + Bte^{rt}$$

and if $r_1 \neq r_2$ the general solution is

$$y = Ae^{r_1 t} + Be^{r_2 t}.$$

In both cases $y \to 0$ as $t \to \infty$. In the second case we have what is called **over damping** and in the first case the over damping is said to be **critical**. In each the mass returns to its equilibrium position without oscillations.

Suppose now that our mass-spring system is subject to an external force so that our DE now becomes

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t).$$

The function $F(t)$ is called the **forcing function** and measures the magnitude and direction of the external force. We consider the important special case where the forcing function is harmonic

$$F(f) = F_0\cos(\gamma t), \quad F_0 > 0 \text{ a constant.}$$

We also assume that we have underdamping with damping constant $b > 0$. In this case the DE has a particular solution of the form

$$y_p = A_1\cos(\gamma t) + A_2\sin(\gamma t).$$

Substituting the the DE and simplifying, we get

$$((k - m\gamma^2)A_1 + b\gamma A_2)\cos(\gamma t) + (-b\gamma A_1 + (k - m\gamma^2)A_2)\sin(\gamma t) = F_0\cos(\gamma t).$$

Setting the corresponding coefficients on both sides equal, we get

$$\begin{aligned} (k - m\gamma^2)A_1 + b\gamma A_2 &= F_0, \\ -b\gamma A_1 + (k - m\gamma^2)A_2 &= 0. \end{aligned}$$

Solving for A_1, A_2 we get

$$A_1 = \frac{F_0(k - m\gamma^2)}{(k - m\gamma^2)^2 + b^2\gamma^2}, \quad A_2 = \frac{F_0 b\gamma}{(k - m\gamma^2)^2 + b^2\gamma^2}$$

and

$$\begin{aligned} y_p &= \frac{F_0}{(k - m\gamma^2)^2 + b^2\gamma^2}((k - m\gamma^2)\cos(\gamma t) + b\gamma\sin(\gamma t)) \\ &= \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}\sin(\gamma t + \phi). \end{aligned}$$

The general solution of our DE is then

$$y = Ce^{-bt/2m}\sin(\omega t + \theta) + \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}\sin(\gamma t + \phi).$$

Because of damping the first term tends to zero and is called the **transient** part of the solution. The second term, the **steady-state** part of the solution, is due to the presence of the forcing function $F_0\cos(\gamma t)$. It is harmonic with the same frequency $\gamma/2\pi$ but is out of phase with it by an angle $\phi - \pi/2$. The ratio of the magnitudes

$$M(\gamma) = \frac{1}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}$$

is called the **gain** factor. The graph of the function $M(\gamma)$ is called the **resonance curve**. It has a maximum of

$$\frac{1}{b\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

when $\gamma = \gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$. The frequency $\gamma_r/2\pi$ is called the **resonance frequency** of the system. When $\gamma = \gamma_r$ the system is said to be in resonance with the external force. Note that $M(\gamma_r)$ gets arbitrarily large as $b \to 0$. We thus see that the system is subject to large oscillations if the damping constant is very small and the forcing function has a frequency near the resonance frequency of the system.

The above applies to a simple LRC electrical circuit where the differential equation for the current I is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + I/C = F(t)$$

where L is the inductance, R is the resistance and C is the capacitance. The resonance phenomenon is the reason why we can send and receive and amplify radio transmissions sent simultaneously but with different frequencies.