McGill University
Math 325A: Differential Equations
Notes for Lecture 11
Text: Ch. 4,6
Linear Differential Equations

In this lecture we will develop the theory of linear differential equations. The starting point is the fundamental existence theorem for the general $n$-th order ODE $L(y) = b(x)$, where

$$L(y) = D^n + a_1(x)D^{n-1} + \cdots + a_n(x).$$

We will also assume that $a_0(x), a_1(x), \ldots, a_n(x), b(x)$ are continous functions on the interval $I$.

**Theorem.** For any $x_0 \in I$, the initial value problem

$$L(y) = b(x) \qquad y(x_0) = d_1, y'(x_0) = d_2, \ldots, y^{(n-1)}(x_0) = d_n$$

has a unique solution for any $d_1, d_2, \ldots, d_n \in \mathbb{R}$.

If $V$ is the solution space of the associated homogeneous DE $L(y) = 0$, the transformation

$$T : V \to \mathbb{R}^n,$$

defined by $T(y) = (y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0))$, is linear transformation of the vector space $V$ into $\mathbb{R}^n$ since

$$T(ay + bz) = aT(y) + bT(z).$$

Moreover, the fundamental theorem says that $T$ is one-to-one ($T(y) = T(z) \implies y = z$) and onto (every $d \in \mathbb{R}^n$ is of the form $T(y)$ for some $y \in V$). A linear transformation which is one-to-one and onto is called an **isomorphism**. Isomorphic vector spaces have the same properties.

**Corollary.** $\dim(V) = n$.

This means that there exists $y_1, y_2, \ldots, y_n \in V$ such that every $y \in V$ can be uniquely written in the form

$$y = c_1y_1 + c_2y_2 + \ldots c_ny_n$$

with $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Such a sequence of elements of a vector space $V$ is called a **basis** for $V$. In the context of DE's it is also known as a **fundamental set**. The number of elements in a basis for $V$ is called the dimension of $V$ and is denoted by $\dim(V)$. If

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$$

is the standard basis of $\mathbb{R}^n$ and $y_i$ is the unique $y_i \in V$ with $T(y_i) = e_i$ then $y_1, y_2, \ldots, y_n$ is a basis for $V$. This follows from the fact that

$$T(c_1y_1 + c_2y_2 + \cdots + c_ny_n) = c_1T(y_1) + c_2T(y_2) + \cdots + c_nT(y_n).$$

A set of vectors $v_1, v_2, \ldots, v_n$ in a vector space $V$ is said to **span** or **generate** $V$ if every $v \in V$ can be written in the form

$$v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

with $c_1, c_2, \ldots, c_n \in \mathbb{R}$. The set

$$\operatorname{span}(v_1, v_2, \ldots, v_n) = \{c_1v_1 + c_2v_2 + \cdots + c_nv_n \mid c_1, c_2, \ldots, c_n \in \mathbb{R}\}$$

consisting of all possible linear combinations of the vectors $v_1, v_2, \dots, v_n$ form a subspace of $V$ called the **span** of $v_1, v_2, \dots, v_n$. Then $V = \mathrm{span}(v_1, v_2, \dots, v_n)$ if and only if $v_1, v_2, \dots, v_n$ spans $V$.

The vectors $v_1, v_2, \dots, v_n$ are said to be **linearly independent** if

$$c_1 v_1 + c_2 v_2 + \dots c_n v_n = 0$$

implies that the scalars $c_1, c_2, \dots, c_n$ are all zero. A basis can also be characterized as a linearly independent generating set since the uniqueness of representation is equivalent to linear independence. More precisely,

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = c_1' v_1 + c_2' v_2 + \dots + c_n' v_n \implies c_i = c_i' \text{ for all } i$$

if and only if $v_1, v_2, \dots, v_n$ are linearly independent.

As an example of a linearly independent set of functions consider

$$\cos(x), \cos(2x), \cos(3x).$$

To prove their linear independence, suppose that $c_1, c_2, c_3$ are scalars such that

$$c_1 \cos(x) + c_2 \cos(2x) + c_3 \sin(3x) = 0$$

for all $x$. Then setting $x = 0, \pi/2, \pi$, we get

$$\begin{aligned} c_1 + c_2 &= 0, \\ -c_2 - c_3 &= 0, \\ -c_1 + c_2 &= 0 \end{aligned}$$

from which $c_1 = c_2 = c_3 = 0$.

An example of a linearly dependent set would be $\sin^2(x), \cos^2(x), \cos(2x)$ since

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

implies that $\cos(2x) + \sin^2(x) + (-1)\cos^2(x) = 0$.

Another criterion for linear independence of functions involves the Wronskian. If $y_1, y_2, \dots, y_n$ are $n$ functions which have derivatives up to order $n-1$ then the Wronskian of these functions is the determinant

$$W = W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

If $W(x_0) \neq 0$ for some point $x_0$, then $y_1, y_2, \dots, y_n$ are linearly independent. This follows from the fact that $W(x_0)$ is the determinant of the coefficient matrix of the linear homogeneous system of equations in $c_1, c_2, \dots, c_n$ obtained from the dependence relation

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0$$

and its first $n-1$ derivatives by setting $x = x_0$.

For example, if $y_1 = \cos(x), \cos(2x), \cos(3x)$ we have

$$W = \begin{vmatrix} \cos(x) & \cos(2x) & \cos(3x) \\ -\sin(x) & -2\sin(2x) & -3\sin(3x) \\ -\cos(x) & -4\cos(2x) & -9\cos(3x) \end{vmatrix}$$

and $W(\pi/4)) = -8$ which implies that $y_1, y_2, y_3$ are linearly independent. Note that $W(0) = 0$ so that you cannot conclude linear dependence from the vanishing of the Wronskian at a point. This is not the case if $y_1, y_2, \ldots, y_n$ are solutions of an $n$-th order linear homogeneous ODE.

**Theorem.** If $y_1, y_2, \ldots, y_n$ are solutions of the linear ODE $L(y) = 0$, the following are equivalent:

1. $y_1, y_2, \ldots, y_n$ is a basis for $\ker(L)$;
2. $y_1, y_2, \ldots, y_n$ are linearly independent;
3. $y_1, y_2, \ldots, y_n$ generate $\ker(L)$;
4. $W(y_1, y_2, \ldots, y_n) \neq 0$ at some point $x_0$;
5. $W(y_1, y_2, \ldots, y_n)$ is never zero.

**Proof.** The equivalence of $1, 2, 3$ follows from the fact that $\ker(L)$ is isomorphic to $\mathbb{R}^n$. The rest of the proof follows from the fact that if the Wronskian were zero at some point $x_0$ the homogeneous system of equations

$$\begin{aligned} c_1 y_1(x_0) + c_1 y_2(x_0) + \cdots + c_n y_n(x_0) &= 0 \\ c_1 y_1'(x_0) + c_1 y_2'(x_0) + \cdots + c_n y_n'(x_0) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x_0) + c_1 y_2^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) &= 0 \end{aligned}$$

would have a non-zero solution for $c_1, c_2, \ldots, c_n$ which would imply that

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

and hence that $y_1, y_2, \ldots, y_n$ are not linearly independent. **QED**

The fact that the Wronskian of $n$ solutions of the $n$-th order linear ODE $L(y) = 0$ is either identically zero or vanishes nowhere also follows from the fact that

$$\frac{dW}{dx} = -a_1(x)W$$

from which

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(t)dt}.$$

From the above we see that to solve the $n$-th order linear DE $L(y) = b(x)$ we first find linear $n$ independent solutions $y_1, y_2, \ldots, y_n$ of $L(y) = 0$. Then, if $y_P$ is a particular solution of $L(y) = b(x)$, the general solution of $L(y) = b(x)$ is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_P.$$

The initial conditions $y(x_0) = d_1, y'(x_0) = d_2, \ldots, y_n^{(n-1)}(x_0) = d_n$ then determine the constants $c_1, c_2, \ldots, c_n$ uniquely.