McGill University
Math 325A: Differential Equations
Notes for Lecture 10
Text: Ch. 4,6
Solving Higher Order Differential Equations

In this lecture we give an introduction to several methods for solving higher order differential equations. Most of what we say will apply to the linear case as there are relatively few non-numerical methods for solving non linear equations. There are two important cases however where the DE can be reduced to one of lower degree. The first is a DE of the form

$$y^{(n)} = f(x, y', y'', \dots, y^{(n-1)})$$

where on the right-hand side the variable y does not appear. In this case, setting $z = y'$ leads to the DE

$$z^{(n-1)} = f(x, z, z', \dots, z^{(n-2)})$$

which is of degree $n-1$. If this can be solved then one obtains y by integration with respect to x.

For example, consider the DE $y'' = (y')^2$. Then, setting $z = y'$, we get the DE $z' = z^2$ which is a separable first order equation for z. Solving it we get $z = -1/(x+C)$ or $z = 0$ from which $y = -\log(x+C) + D$ or $y = C$. The reader will easily verify that there is exacly one of these solutions which satisfies the initial condition $y(x_0) = y_0$, $y'(x_0) = y_0'$ for any choice of x_0, y_0, y_0' which confirms that it is the general solution since the fundamental theorem guarantees a unique solution.

The second case is a DE of the form $y^{(n)} = f(y, y', y'', \dots, y^{(n-1)})$ where the independent variable x does not appear explicitly on the right-hand side of the equation. Here we again set $z = y'$ but try for a solution z as a fuction of y. Then, using the fact that $\frac{d}{dx} = z\frac{d}{dy}$, we get the DE

$$(z\frac{d}{dy})^{n-1}(z) = f(y, z, z\frac{dz}{dy}, \dots, (z\frac{d}{dy})^n(z))$$

which is of degree $n-1$. For example, the DE $y'' = (y')^2$ is of this type and we get the DE

$$z\frac{dz}{dy} = z^2$$

which has the solution $z = Ce^y$. Hence $y' = Ce^y$ from which $-e^{-y} = Cx + D$. This gives $y = -\log(-Cx - D)$ as the general solution which is in agreement with what we did previously.

Let us now go to linear equations. The general form is

$$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x).$$

The function L is called a differential operator. The characteristic feature of L is that

$$L(a_1y_1 + a_2y_)) = a_1L(y_1) + a_2L(y_2).$$

Such a function L is what we call a linear operator. This linearity implies that for any two solutions y_1, y_2 the difference $y_1 - y_2$ is a solution of the associated homogeneous equation $L(y) = 0$. Morerover, it implies that any linear combination $a_1y_1 + a_2y_2$ of solutions y_1, y_2 of $L(y) = 0$ is again a solution of $L(y) = 0$. The solution space of $L(y) = 0$ is also called the **kernel** of L and is denoted by $\ker(L)$.

It is a subspace of the vector space of real valued functions on some interval I. If y_p is a particular solution of $L(y) = b(x)$, the general solution of $L(y) = b(x)$ is

$$\ker(L) + y_p = \{y + y_p \mid L(y) = 0\}.$$

As an example consider the linear DE $y'' + 2y' + y = x$. Here $L(y) = y''2y' + y$. A particular solution of the DE $L(y) = x$ is $y_p = x - 2$. The associated homogeneous equation is

$$y'' + 2y'' + y = 0.$$

We will give several methods for solving this DE. The first is based on finding one solution. In this case one can easily discover that e^{-x} is a solution. For linear equations, there is a method of reduction of order which guarantees the existence of a solution of the form $C(x)e^{-x}$ with $C(x)$ satisfying a linear DE of lower order. To see this in this case substitute $y = C(x)e^{-x}$ in the equation $y'' - 2y' + y = 0$ to get $C''(x)e^{-x} = 0$ and hence $C(x) = Ax + B$. Thus

$$Axe^{-x} + Be^{-x}$$

is a two parameter family of solutions consisting of the linear combinations of the two solutions $y_1 = xe^{-x}$ and $y_2 = e^{-x}$. That it is the general solution we make use of the fundamental theorem which states that if y, z are two solutions such that $y(0) - z(0)$ and $y'(0) = z'(0)$ then $y = z$. Let y be any solution and consider the linear equations in A, B

$$\begin{aligned} Ay_1(0) + By_2(0) &= y(0), \\ Ay_1'(0) + By_2'(0) &= y'(0). \end{aligned}$$

They have the unique solution $A = y'(0) - y(0)$, $B = y(0)$. With this choice of A, B the solution $z = Ay_1 + By_2$ satisfies $z(0) = y(0)$, $z'(0) = y'(0)$ and hence $y = z$. Thus the general solution of the DE

$$y'' + 2y' + y = x$$

is $y = Axe^{-x} + Be^{-x} + x - 2$.

This equation can be solved quite simply without the use of the fundamental theorem if we make essential use of operators. The differential operator $L(y) = y'$ is denoted by D. The operator $L(y) = y''$ is nothing but $D^2 = D \circ D$ where $\circ$ denotes composition of functions. More generally, the operator $L(y) = y^{(n)}$ is D^n. If

$$L_1(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y$$

$$L_2(y) = b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_n(x)y$$

and $p_1(x), p_2(x)$ are functions of x the function $p_1L_1 + p_2L_2$ defined by

$$\begin{aligned} (p_1L_1 + p_2L_2)(y) &= p_1(x)L_1(y) + p_2(x)L_2(y) \\ &= (a_0(x) + p_2(x)b_0(x)y^{(n)} + \cdots (p_1(x)a_n(x) + p_2(x)b_n(x))y \end{aligned}$$

is again a linear differential operator. An important property of linear operators in general is the distributive law

$$L(L_1 + L_2) = LL_1 + LL_2, \quad (L_1 + L_2)L = L_1L + L_2L.$$

The identity operator I is defined by $I(y) = y$. By definition $D^0 = I$. The general linear n-th order ODE can therefore be written

$$(a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_n(x)I)(y) = b(x).$$

The DE $y'' + 2y' + y = x$ can therefore be written in operator form as

$$(D^2 + 2D + I)(y) = x.$$

The operator $D^2 + 2D + I$ can be factored as $(D + I)^2$. We now make use of the fact that, for any scalar a,

$$D - a = e^{ax} D e^{-ax}$$

where the factors e^{ax}, e^{-ax} are interpreted as linear operators. This identity is just the fact that

$$\frac{dy}{dx} - ay = e^{ax}(\frac{d}{dx}(e^{-ax} y).$$

Hence $(D + I)^2 = e^{-x} D e^{x} e^{-x} D e^{x} = e^{-x} D^2 e^{x}$. and so the DE $(D + I)^2(y) = x$ can be written $e^{-x} D^2 e^{x}(y) = x$ or

$$\frac{d^2}{dx}(e^x y) = xe^{-x}$$

which on integrating twice gives

$$e^x y = xe^{-x} - 2e^{-x} + Ax + B, \quad y = x - 2 + Axe^{-x} + Be^{-x}.$$

We leave it to the reader to prove that

$$\ker((D - a)^n) = \{(a_0 + a_x + \cdots + a_{n-1}x^{n-1})e^{ax} \mid a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}\}.$$

Now consider the DE $y'' - 3y' + 2y = e^x$. In operator form this equation is

$$(D^2 - 3D + 2I)(y) = e^x.$$

Since $(D^2 - 3D + 2I = (D - I)(D - 2I)$ this DE can be written

$$(D - I)(D - 2I)(y) = e^x.$$

Now let $z = (D - 2I)(y)$. Then $(D - I)(z) = e^x$, a first order linear DE which has the solution $z = xe^x + Ae^x$. Now $z = (D - 2I)(y)$ is the linear first order DE

$$y' - 2y = xe^x + Ae^x$$

which has the solution $y = e^x - xe^x - Ae^x + Be^{2x}$. Notice that $-Ae^x + Be^{2x}$ is the general solution of the associated homogeneous DE $y'' - 3y' + 2y = 0$ and that $e^x - xe^x$ is a particular solution of the original DE $y'' - 3y' + 2y = e^x$. We leave to the reader the proof of the fact that for $a \neq b$

$$\ker((D - a)(D - b)) = \{Ae^{ax} + Be^{bx} \mid A, B \in \mathbb{R}\}.$$

and that for any a, b

$$\ker((D + a)^2 + b^2) = \{Ae^{-ax}\cos(bx) + Be^{-bx}\sin(bx) \mid A, B \in \mathbb{R}\}.$$

As an example of the use of this consider the DE

$$y'' + 2y' + 5y = \sin(x)$$

which in operator form is $(D^2 + 2D + 5I)(y) = \sin(x)$. Now

$$D^2 + 2D + 5I = (D + I)^2 + 4I$$

and so the associated homogeneous DE has the general solution

$$Ae^{-x}\cos(2x) + Be^{-x}\sin(2x).$$

All that remains is to find a particular solution of the original DE. We leave it to the reader to show that there is a particular solution of the form $C\cos(x) + D\sin(x)$.