McGill University
Math 325A: Differential Equations
Notes for Lecture 1

Text: Sections 1.1, 1.2

An **Ordinary Differential Equation (ODE)** is an equation involving the derivatives of an unknown function $y$ of a single variable $x$. Any function $y = f(x)$ which satisfies this equation is called a solution of the ODE. For example, $y = e^{2x}$ is a solution of the ODE

$$y' = 2y$$

and $y = \sin(x^2)$ is a solution of the ODE

$$xy'' - y' + 4x^3y = 0.$$

An ODE is said to be of **order** $n$ if $y^{(n)}$ is the highest order derivative occurring in the equation. The simplest first order ODE is $y' = g(x)$. The most general form of an $n$-th order ODE is

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

with $F$ a function of $n+2$ variables $x, u_0, u_1, \ldots, u_n$. The equations

$$xy'' + y = x^3, \quad y' + y^2 = 0, \quad y''' + 2y' + y = 0$$

are examples of ODE's of second order, first order and third order respectively with respectively

$$F(x, u_0, u_1, u_2) = xu_2 + u_0 - x^3, \quad F(x, u_0, u_1) = u_1 + u_0^2, \quad F(x, u_0, u_1, u_2, u_3) = u_3 + 2u_1 + u_0.$$

If the function $F$ is linear in the variables $u_0, u_1, \ldots, u_n$ the ODE is said to be **linear**. If, in addition, $F$ is homogeneous then the ODE is said to be homogeneous. The first of the above examples above is linear are linear, the second is non-linear and the third is linear and homogeneous. The general $n$-th order linear ODE can be written

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

This DE is homogeneous if and only if $b(x) \equiv 0$. Linear homogeneous equations have the important property that linear combinations of solutions are also solutions. In other words, if $y_1, y_2, \ldots, y_m$ are solutions and $c_1, c_2, \ldots, c_m$ are constants then

$$c_1y_1 + c_2y_2 + \cdots + c_my_m$$

is also a solution.

A **Partial Differential Equation (PDE)** is an equation involving the partial derivatives of a function of more than one variable. The concepts of linearity and homogeneity can be extended to PDE's. The general second order linear PDE in two variables $x, y$ is

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y).$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a linear, homogeneous PDE of order 2. The functions $u = \log(x^2 + y^2)$, $u = xy$, $u = x^2 - y^2$ are examples of solutions of Laplace's equation. We will not study PDE's systematically in this course.

By the **general solution** of a differential equation we mean the set of all solutions, i.e., the set of all functions which satisfy the equation. For example, the general solution of the differential equation $y' = 3x^2$ is $y = x^3 + C$ where $C$ is an arbitrary constant. The constant $C$ is the value of $y$ at $x = 0$. This **initial condition** completely determines the solution. More generally, one easily shows that given $a, b$ there is a unique solution $y$ of the differential equation with $y(a) = b$. Geometrically, this means that the one-parameter family of curves $y = x^2 + C$ do not intersect one another and they fill up the plane $\mathbb{R}^2$.

An $n$-th order ODE of the form $y^{(n)} = G(x, y, y', \ldots, y^{n-1})$ is said to be in **normal** form. If we introduce dependant variables $y_1 = y, y_2 = y', \ldots, y_n = y^{n-1}$ we obtain the equivalent system of first order equations

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_n' &= G(x, y_1, y_2, \ldots, y_n). \end{aligned}$$

For example, the ODE $y'' = y$ is equivalent to the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_1. \end{aligned}$$

In this way the study of $n$-th order equations can be reduced to the study of systems of first order equations. Systems of equations arise in the study of the motion of particles. For example, if $P(x, y)$ is the position of a particle of mass $m$ at time $t$, moving in a plane under the action of the force field $(f(x, y), g(x, y)$, we have

$$\begin{aligned} m\frac{d^2x}{dt^2} &= f(x, y), \\ m\frac{d^2y}{dt^2} &= g(x, y). \end{aligned}$$

This is a second order system.

The general first order ODE in normal form is

$$y' = F(x, y).$$

If $F$ and $\frac{\partial F}{\partial y}$ are continuous one can show that, given $a, b$, there is a unique solution with $y(a) = b$. Describing this solution is not an easy task and there are a variety of ways to do this. The dependence of the solution on initial conditions is also an important question as the initial values may be only known approximately.

The non-linear ODE $yy' = 4x$ is not in normal form but can be brought to normal form

$$y' = \frac{4x}{y}.$$

by dividing both sides by $y$. No solutions are lost since no solution satisfies $y(x) = 0$ if $x \neq 0$. Here $F(x, y) = 4x/y$ is not continuous on the line $y = 0$ so that the above existence and uniqueness theorem does not apply if $b = 0$; in fact, there is no solution with $y(a) = 0$ if $a \neq 0$. If we integrate both sides of the original DE and simplify we get the one-parameter family

$$y^2 = 4x^2 + C$$

which defines $y$ implicitly as a function of $x$. If $C = 0$, we have $y = \pm x$ so that there are two solutions $y = x$ and $y = -x$ passing through $(0, 0)$.

If $(a, b)$ is a point on the curve $f(x, y) = C$ with $\frac{\partial f}{\partial y}(a, b) \neq 0$, there is a function $y = \phi(x)$ with $\phi(a) = b$ and $f(x, \phi(x)) = 0$ for all $x$ sufficiently near $a$. Differentiating $f(x, y) = C$ implicitly with respect to $x$, we get

$$y' = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial x}}$$

for all $x, y$ sufficiently near $(a, b)$. The solutions of this DE satisfy the equation $f(x, y)$=C.