McGill University
Math 319B: Partial Differential Equations

Assignment 5 Solutions

1. We first separate off the t variable by looking for solutions of the form $u = T(t)v(x, y)$. This gives $T' = \lambda T$ and $\nabla^2(v) = \lambda v$ with $v = 0$ on $x^2+y^2 = 1$ and v a function of $r = \sqrt{x^2 + y^2}$. Using the results proved in class, it follows that only the eigenfunctions $\phi_n(r) = J_0(z_{0n}r)$ with eigenvalue $\lambda_n = -z_{0n}^2$ will be required for the solution. By the superposition principle, we look for a solution of the form

$$v(x, y, t) = \sum_{n=0}^{\infty} a_n\phi_n(r)e^{-z_{0n}^2 t}.$$

Using the initial condition we get

$$f(r) = v(x, y, 0) = \sum_{n=0}^{\infty} a_n\phi_n(r)$$

so that the coefficients a_n are given by

$$a_n = \frac{\int_0^1 f(r)\phi_n(r)r\,dr}{\int_0^1 \phi(r)^2 r\,dr}.$$

The function $u(x, y, t)$ give the temperature at time t at a point (x, y) of a thin circular plate of radius 1 with both sides insulated, with initial temperature $f(\sqrt{x^2 + y^2})$ at (x, y) and circular edge maintained at zero degrees for $t > 0$.

2. If $u'' = -\lambda u$ with $u \neq 0$, we have

$$-\lambda \int_0^1 u^2\,dx = \int_0^1 uu''\,dx = uu'|_0^1 - \int_0^1 (u')^2 dx = -u(1)^2 - \int_0^1 (u')^2 dx$$

which shows that $\lambda > 0$. If $\lambda = \mu^2$, $\mu > 0$, we have

$$u(x) = Asin(\mu x) + B\cos(x)$$

and $u(0) = 1$ implies $A = 0$. Hence $u(x) = B\sin(\mu x)$ with $B \neq 0$. Now $u'(1) = -u(1)$ gives

$$B\mu\cos(\mu) = -B\sin(\mu)$$

and hence $\tan(\mu) = -\mu$. The solutions $\mu > 0$ of the equation are μ_n $(n \geq 1)$ with $(2n-1)\pi/2 < \mu_n < (2n+1)\pi/2$ and $\mu_n \approx (2n-1)\pi/2$. To prove the orthogonality of the eigenfunctions $\phi_n(x) = \sin(\mu_n x)$ with eigenvalue $\lambda_n = -\mu_n^2$, we use the inner product

$$< \phi, \psi >= \int_0^1 \phi(x)\psi(x)\,dx.$$

If $n \neq m$, we have $\lambda_n \neq \lambda_m$ so that

$$\lambda_n < \phi_n, \phi_m >=< \lambda_n\phi_n, \phi_m >=< -\phi_n'', \phi_m >=< \phi_n, -\phi_m'' >=< \phi_n, \lambda_m\phi_m >= \lambda_m < \phi_n, \phi_m >$$

which implies $< \phi_n, \phi_m >= 0$. The fact that $< \phi_n'', \phi_m >=< \phi_n, \phi_m'' >$ follow from

$$\int_0^1 \phi_n''\phi_m dx = (\phi_n'\phi_m - \phi_m'\phi_n)|_0^1 + \int_0^1 \phi_n\phi_m'' dx$$

and

$$(\phi_n'\phi_m - \phi_m'\phi_n)|_0^1 = -\phi_n(1)\phi_m(1) + \phi_m(1)\phi_n(1) = 0.$$