McGill University
Math 319B: Partial Differential Equations

Assignment 4 Solutions

1. (a) We have $f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$ with $A_0 = \int_0^1 (1-x)\,dx = 1/2$,

$$A_n = 2\int_0^1 (1-x)\cos(n\pi x) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 4/n^2\pi^2 & \text{if } n \text{ is odd.} \end{cases}$$

(b) The even periodic extension of $f(x) = 1 - x$ is the function $F(x)$ defined by

$$\begin{cases} 1-x & \text{if } 0 \le x \le 1, \\ 1+x & \text{if } -1 \le x \le 0, \\ F(x) = F(x+2) & \text{for all } x \in \mathbb{R}. \end{cases}$$

Since $F(x)$ is continuous, piecewise smooth, and the Fourier series of $F(x)$ is the Fourier cosine series of $f(x)$, we have by Fourier's Theorem

$$F(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{k=0}^{\infty} \frac{\cos((2k+1)\pi x)}{(2k+1)^2}.$$

(c) Since $F(x)$ is continuous and piecewise smooth, we have

$$F'(x) \sim -\frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}.$$

Since $F'(x)$ is piecewise smooth the series converges for all x and we have equality at all point where $F''(x)$ exists, namely, for x not an integer. The series is zero if x is an integer.

2. (a) We have $f(x) \sim \sum_{n=1}^{\infty} A_n \sin(n\pi x)$ with $A_n = 2\int_0^1 (1-x)\sin(n\pi x) = 2/\pi$.

(b) The odd periodic extension of $f(x)$ is the function $F(x)$ defined by

$$\begin{cases} 1-x & \text{if } 0 < x < 1, \\ -1-x & \text{if } -1 < x < 0, \\ 0 & \text{if } x \text{ is an integer,} \\ F(x) = F(x+2) & \text{for all } x \in \mathbb{R}. \end{cases}$$

Since $F(x)$ is piecewise smooth, and the Fourier series of $F(x)$ is the Fourier sine series of $f(x)$, we have by Fourier's Theorem

$$F(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}$$

for all x since $(F(x+) + F(x-))/2 = F(x)$ for all x.

(c) The term by term derivative of the above Fourier series is $\sum_{n=1}^{\infty} 2\cos(n\pi x)$ which diverges for all x.

(d) Since $f'(x) = -1$, the Fourier cosine series of $f'(x)$ is -1.

3. (a) We have $f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) + B_n \sin(nx)$ with

$$A_0 = (1/2\pi)\int_{-\pi}^{\pi} f(x) = (1/2\pi)\int_{0}^{\pi} \sin(x)\,dx = 1/\pi,$$

$$A_n = (1/\pi)\int_{0}^{\pi} \sin(x)\cos(nx)\,dx = \begin{cases} 0 & \text{if } n \text{ odd,} \\ \frac{2}{\pi}\frac{1}{1-n^2} & \text{if } n \text{ even,} \end{cases}$$

$$B_n = (1/\pi)\int_{0}^{\pi} \sin(x)\sin(nx)\,dx = \begin{cases} \frac{1}{2} & \text{if } n = 1, \\ 0 & \text{if } n \neq 1, \end{cases}$$

Since the periodic extension $F(x)$ of $f(x)$ of period 2π is continuous and piecewise smooth and the Fourier series of $F(x)$ is the Fourier series of $f(x)$ we have

$$F(x) = \frac{1}{\pi} + \frac{\sin(x)}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2kx)}{1-4k^2}.$$

(b) Since $F'(x)$ is piecewise smooth the Fourier series of $F'(x)$ converges for all x and is the series obtained by differentiating the Fourier series of $F(x)$ term by term. Since $F'(x)$ is continuous for $x \neq k\pi$

$$F'(x) = \frac{\cos(x)}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{2n}{1-4n^2}\sin(2nx)$$

for $x \neq k\pi$. If $G(x)$ is the function defined by Fourier series of $F'(x)$, we have $G(x) = (-1)^k/2$ if $x = k\pi$ and $G(x) = F'(x)$ otherwise.

4. (a) If $F(x)$ is the odd periodic extension of $f(x) = 1 - x$ on $0 \leq x \leq 1$, we have

$$F(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\pi x)}{n}.$$

Integrating both sides from 0 to x, we get

$$G(x) = \int_{0}^{x} F(x)\,dx = \frac{2}{\pi^2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(n\pi x)}{n^2}.$$

We have

$$G(x) = \begin{cases} x - x^2/2 & \text{if } 0 \leq x \leq 1, \\ -x + x^2/2 & \text{if } -1 \leq x \leq 0, \\ G(x+2) = G(x) & \text{for all } x. \end{cases}$$