McGill University
Math 319B: Partial Differential Equations

Solutions For Assignment 2

1. The general solution of the PDE is $u(x,t) = f(x-ct) + g(x+ct)$. Then $u(x,t) = e^x$, $\frac{\partial u}{\partial t}(x,0) = \sin(x)$ give
$$f(x) + g(x) = e^x, \quad -cf'(x) + cg'(t) = \sin(x)$$
Differentiating the first equation and adding $c$ times it to the second gives
$$2cg'(x) = \sin(x) + ce^x$$
from which $g'(x) = (1/2c)\sin(x) + e^x$ and hence
$$g(x) = -\frac{1}{2c}\cos(x) + \frac{1}{2}e^x + A, \quad f(x) = \frac{1}{2}e^x + \frac{1}{2c}\cos(x) - A$$
from which $u(x,t) = (1/2)e^{x-ct} + (1/2)e^{x+ct} + (1/2c)\cos(x-ct) - (1/2)\cos(x+ct)$.

2. **Method 1:** Setting $X = \frac{\partial}{\partial x}$, $T = \frac{\partial}{\partial t}$ the given PDE can be written
$$(X^2 - 3XT - 4T^2)u = (X+T)(X-4T)u = 0.$$
Setting $v = (X-4T)u$, we have $(X+T)v = 0$. The solution of the latter PDE is $v(x,y) = f(t-x)$. Substituting this in $(X-4T)u = v$ we get
$$\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial t} = f(t-x).$$
The solution of this PDE reduces to solving
$$\frac{dt}{dx} = -4, \quad \frac{du}{dx} = f(t-x)$$
which yields $u(x,t) = g(t-x) + h(t+4x)$. Using the initial conditions
$$u(x,0) = x^2, \quad \frac{\partial u}{\partial t}(x,0) = e^x,$$
we get $g(-x) + h(4x) = x^2$, $g'(-x) + h'(4x) = e^x$. Differentiating the first equation and adding it to the second gives
$$5h'(4x) = 2x + e^x$$
from which $h(x) = x^2/20 + 4e^{x/4}/5 + C$, $g(x) = x^2/5 - 4e^{-x}/5 - C$ and
$$u(x,t) = \frac{1}{5}(t-x)^2 + \frac{1}{20}(t+4x)^2 - \frac{4}{5}e^{x-t} + \frac{4}{5}e^{(t+4x)/4}.$$
**Method 2:** Completing squares in $X$, we have
$$X^2 - 3XT - 4T^2 = (X - 3T/2)^2 - 25T^2/4 = \frac{\partial^2}{\xi^2} - \frac{\partial^2}{\partial\tau^2},$$

where $x = \xi$, $t = -(3/2)\xi + (5/2)\tau$. The general solution of the PDE

$$\frac{\partial u^2}{\partial \xi^2} - \frac{\partial u^2}{\partial \tau^2} = 0$$

is $u = G(\tau - \xi) + H(\tau + \xi)$, where $\xi = x$, $\tau = (3/5) + (2/5)t$. This gives

$$u = G(2(t - x)/5) + H(2(t + 4x)/5) = g(t - x) + h(t + 4x)$$

and the rest of the solution is the same as for method 1.

3. Setting $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$, we have

$$X^2 + 3Y^2 - 2X + 24Y + 5 = (X - 1)^2 + 3(Y - 4)^3 - 44 = e^x X^2 e^{-x} + 3e^{4y}e^{-4} - 44$$

so that setting $v = e^{-x-4y}u$, the given PDE becomes

$$\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial^2 v}{\partial y^2} - 44v = 0$$

which, on setting $x = \xi$, $y = \sqrt{3}\eta$, becomes

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - 44v = 0.$$

4. Using the same notation as in question 3, we have

$$X^2 - 4XY + 4Y^2 + 2X + 1 = (X - 2Y)^2 + 2X + 1 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial}{\partial \eta} + 1$$

where $\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}$. If we now let $v = e^{\eta}u$, the given PDE becomes

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial v}{\partial \eta} = 0.$$

5. Using the same notation as in the previous problem, we have

$$\begin{aligned} X^2 - 4XY + 4Y - 1 &= (X - 2Y)^2 - 4Y^2 + 4Y - 1 \\ &= \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} + 2\frac{\partial}{\partial \eta} - 1 = \frac{\partial^2}{\partial \xi^2} - (\frac{\partial}{\partial \eta} - 1)^2 \end{aligned}$$

where $\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$, $\frac{\partial}{\partial \eta} = 2\frac{\partial}{\partial y}$. Setting $v = e^{-\eta}u$, the given PDE becomes

$$\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \eta^2} = 0.$$

The general solution of this PDE is $v = f(\eta - \xi) + g(\eta + \xi)$. Using $u = e^{\eta}v$ and $\xi = x$, $\eta = x + y/2$ we obtain

$$u = e^{x+y/2}(f(y/2) + g(2x + y/2)) = e^{x+y/2}(F(y) + G(y + 4x))$$

as the general solution.