McGill University
Math 319B: Partial Differential Equations

Solutions For Assignment 1

1. (a) L is linear since

$$L(au+bv) = \frac{\partial}{\partial x}(au+bv) + x\frac{\partial}{\partial y}(au+bv) = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial x} + ax\frac{\partial v}{\partial y} + bx\frac{\partial v}{\partial y}$$
$$= a(\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}) + b(\frac{\partial v}{\partial x} + x\frac{\partial v}{\partial y}) = aL(u) + bL(v);$$

(b) L is not linear since $L(2y) = 4y \neq 2y = 2L(y)$;

(c) L is not linear since $L(2y) = 4 \neq 2 = 2L(y)$;

(d) L is not linear since $L(0+0) = L(0) = 1 \neq 2 = L(0) + L(0)$.

2. (a) The solutions of the PDE are obtained from the solutions of the system

$$\frac{dy}{dx} = \frac{3}{2}, \quad \frac{dz}{dx} = \frac{4}{2} = 2.$$

Solving the first DE, we get $y = 3x/2 + C$. Then $z = u(x,y)$ is a function of x with

$$\frac{dz}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 2$$

and hence $z = 2x + D$ with $D = f(C)$. Hence the general solution of the given PDE is

$$z = u(x,y) = 2x + f(y - 3x/2).$$

(b) We have $u(0,y) = y^2 \iff f(y) = y^2$ so that $u(x,y) = 2x + (y - 3x/2)^2$. This surface can be best visualized by noticing that it is a ruled surface, made up by the 1-parameter family of lines

$$x = t,\ y = 3t/2 + s,\ z = 2t + s^2.$$

For a given s, the line is the characteristic curve passing through $(0, s, s^2)$. It has the constant direction vector $(1, 3/2, 2)$.

3. (a) The characteristic curves are the solutions of the system

$$\frac{dy}{dx} = 2,\ \frac{dz}{dx} = e^{x+2y} - z.$$

Solving the first DE gives $y = 2x + C$. Substituting this into the second DE gives

$$\frac{dz}{dx} + z = e^{5x+2C}.$$

Solving this linear ODE we get $z = e^{5x+2C}/6 + De^{-x}$. Thus the characteristic curves are given by

$$y = 2x + C,\ z = e^{5x+2C}/6 + De^{-x}.$$

(b) The general solution of the given PDE is obtained by setting $D = f(C)$ with $C = y - 2x$. This gives

$$u(x, y) = e^{x+2y}/6 + e^{-x} f(y - 2x).$$

(c) If $u(x, 0) = 0$, we get $u(x, 0) = e^x/6 + e^{-x} f(-2x) = 0$ which gives $f(-2x) = -e^{2x}/6$ and hence $f(x) = -e^{-x}/6$. So $u(x, y) = e^{x+2y}/6 - e^{x-y}/6$.

4. The solution $z = u(x, y)$ is obtained by solving the system

$$\frac{dy}{dx} = \frac{1}{x^2 + 1}, \quad \frac{dz}{dx} = 0.$$

Solving the first gives $y = \arctan(x) + C$ and solving the second yields $z = D$. The general solution is obtained by setting $D = f(C)$ with $C = y - \arctan(x)$. This gives

$$z = u(x, y) = f(y - \arctan(x)).$$

Given $u(0, y) = \sin(y)$, we get $f(y) = \sin(y)$ and hence $u(x, y) = \sin(y - \arctan(x))$.

5. The solutions of th PDE are obtained from the solutions of the system

$$\frac{dx}{dt} = 5t, \quad \frac{du}{dt} = 3u.$$

Solving the first gives $x = 5t^2/2 + C$ and, solving the second, we get $u = De^{3t}$. The general solution of the given PDE is obtained by setting $D = f(C)$ with $C = x - 5t^2/2$. This gives

$$u(x, t) = f(x - 5t^2/2)e^{3t}.$$

Note that $f(x) = u(x, 0)$.

6. (a) We have to solve the system

$$\frac{dy}{dx} = -u^2, \quad \frac{du}{dx} = 3u.$$

Solving the second, we get $u = Ce^{3x}$. Substituting this in the first DE, we get

$$\frac{dy}{dx} = -C^2 e^{6x}.$$

Solving this DE, we get $y = -C^2 e^{6x}/6 + D = -u^2/6 + D$. Setting $D = f(C)$ with $C = ue^{-3x}$ we get

$$y = -u^2/6 + f(ue^{-3x})$$

as the general solution in implicit form.

(b) If $u(0, y) = y$ we get $y = -y^2/6 + f(y)$ so that $f(y) = y + y^2/6$ and

$$y = -u^2/6 + ue^{-3x} + u^2 e^{-6x}/6.$$