McGill University
Math 319B: Partial Differential Equations
Linear First Order PDE's

The general first order linear PDE in two independent variables x, y and one dependent variable $z = u(x,y)$ is

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} + c(x,y)z = d(x,y).$$

This PDE is said to be **homogeneous** if $d(x,y)$ is the zero function. We want to give a procedure for findingxs all functions $u(x,y)$ such that $z = u(x,y)$ satisfies this differential equation. Geometrically, any solution $z = u(x,y)$ is a surface.

Example 1. We begin with the simplest case $\frac{\partial z}{\partial x} = 0$. The function $z = u(x,y)$ is a solution if and only if $z = D$, a constant, on the line $y = C$, with C any arbitrary constant. The constant D depends on C so we have $D = f(C)$, where f can be an arbitrary function of one variable. The general solution of the PDE is therefore $u(x,y) = f(y)$ with f an arbitrary function. Setting $x = 0$, we see that $f(y) = u(0,y)$. Thus there is a unique solution with a prescribed boundary condition $f(0,y) = g(y)$. The surface $z = f(y)$ consists of the family of lines $y = C,\ z = f(C)$. Such a surface is called a **ruled surface**.

Example 2. As a second example, consider the PDE

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

The left-hand side of the equation is a constant multiple of the directional derivative of z in the direction of the vector $(1,1)$. Since the right-hand side of the equation is zero we see that any solution $z = u(x,y)$ must be a constant D on the line $y = x + C$. Again, D must be a function of C, say $D = u(C)$. Thus $u(x,y) = u(y-x)$ and the surface $z = u(x,y)$ consists of the family of lines $z = f(C),\ y = x + C$. Moreover, $f(y) = u(0,y)$ which means that f must be differentiable. Conversely, if f is a diferentiable function of one variable, then $z = f(y-x)$ is a solution of the PDE as one easily verifies.

Example 3. Using a similar argument one can show that the general solution of the PDE

$$\frac{\partial z}{\partial x} + c\frac{\partial z}{\partial x} = 0$$

is $z = u(y - cx)$ with f an arbitrary differentable function of one variable. The surface $z = f(y - cx)$ consists of the lines $z = f(C),\ y - cx = C$. For any fixed x, the graph of $z = f(y - cx)$ is the translation of the graph of $z = f(y)$ by cx units to the right if $c > 0$, to the left if $c < 0$.

We proceed in a similar manner for the homogeneous linear PDE

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = 0.$$

Geometrically, this equation says that any solution $z = f(x,y)$ is a constant D on any curve $\phi(x,y) = C$ whose tangent vector at (x,y) has the same direction as the vector $(a(x,y), b(x,y)$. If $a(x,y) \neq 0$, then the slope of the tangent line to this curve at (x,y) satisfies the ODE

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

If $h(x,y) = b(x,y)/a(x,y)$ is continuous and $\frac{\partial h}{\partial y}$ is also, then this DE has the general solution $y = h(x,C)$ and it can be shown that the solutions can be written in the form $\phi(x,y) = C$. The general solution of the PDE is therefore $z = u(x,y) = f(\phi(x,y))$ with f an arbitrary differentiable function of one variable. The surface $z = u(x,y)$ is formed by the curves $\phi(x,y) = C$, $z = f(C)$.

Example 4. As an example consider the PDE

$$e^x \frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

A solution $z = u(x,y)$ will be constant on the solution curves of

$$\frac{dy}{dx} = xe^{-x}.$$

The general solution of this DE is

$$y = xe^{-x} + e^{-x} + C$$

so that the general solution of the PDE is

$$z = u(x,y) = f(C) = f(y - xe^{-x} - e^{-x})$$

with f an arbitrary differentiable function of one variable. Note that $u(0,y) = f(y-1)$ so that $f(y) = u(0,y+1)$.

Example 5. To solve the PDE

$$-y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

we first solve the ODE

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This ODE has the general solution $x^2 + y^2 = C$ so that the general solution of the given PDE is $z = f(x^2 + y^2)$. This surface is a surface of revolution obtained by revolving the curve $z = f(y)$ about the z-axis

We now consider the more general quasi-linear PDE

$$a(x,y)\frac{\partial z}{\partial x} + b(x,y)\frac{\partial z}{\partial y} = c(x,y,z).$$

We first find the general solution $y = g(x,C)$ of the ODE

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

and write the solution in the form $\phi(x,y) = C$. If $z = u(x,y)$ is a solution of the given PDE then $z = u(x,y)$ is a function of x on the curve $y = g(x,C)$. Hence, by the chain rule,

$$\frac{dz}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{c(x,y,z)}{a(x,y}.$$

If $z = u(x,y) = h(x,C,D)$ is the general solution of this ODE then, since $D = f(C)$ for some function f, we have

$$z = h(x, \phi(x,y), f(\phi(x,y)).$$

The surface $z = u(x, y)$ is composed of curves of the form

$$y = g(x, C), \;\; z = h(x, C, D)$$

which are the solutions of the system of ODE's

$$\begin{aligned}\frac{dy}{dx} &= \frac{b(x,y)}{a(x,y)} \\ \frac{dz}{dx} &= \frac{c(z,xy)}{a(x,y)}.\end{aligned}$$

These curves are called **characteristic curves** of the given PDE. If a, b, c are continuously differentiable in some region of 3-space and $a(x, y) \neq 0$ there, then there is a unique characteristic curve passing through any point of R.

Conversely, if $u(x, y) = h(x, \phi(x, y), f(\phi(x, y))$, then on the curve $\phi(x, y) = C$ we have $u(x, y) = h(x, C, f(C)$ so that $z = u(x, y)$ satisfies

$$\frac{dz}{dx} = \frac{c(z,xy)}{a(x,y)}.$$

and

$$\frac{dz}{dx} = \frac{\partial u}{\partial x} + \partial u \partial y \frac{dy}{dx}$$

with

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

It follows that

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{b(x,y)}{a(x,y)} = \frac{c(z,xy)}{a(x,y)}$$

which shows the $z = u(x, y)$ satisfies the given PDE.

Example 6. The characteristics curves of the PDE

$$x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = nz,$$

where $n \geq 0$ is an integer are the solutions of the system

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x} \\ \frac{dz}{dx} &= n\frac{z}{x}.\end{aligned}$$

The general solution of this system is $y = Cx$, $z = Dx^n$. The general solution of the PDE is therefore

$$z = u(x, y) = x^n f(y/x)$$

with $f(y) = u(1, y)$. The function $u(x, y)$ is homogeneous of degree n, i.e.,

$$u(tx, ty) = t^n u(x, y).$$

Conversely, if $u(x,y)$ is homogeneous of degree n then, on differentiating the above equation with respect to t, we find

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nt^{n-1}u(x,y).$$

If we set $t=1$ we obtain that $z=u(x,y)$ satisfies the PDE

$$x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = nz.$$

If we now allow the functions a and b to be functions of x,y,z we get the general **quasi-linear** PDE

$$a(x,y,z)\frac{\partial z}{\partial x} + b(x,y,z)\frac{\partial z}{\partial y} = c(x,y,z).$$

The above method of characteristic curves extends to these PDE's but the system of ODE's which define these curves is more difficult in general. The general solution of this system is the two-parameter family of curves

$$y = y(x,C,D),\ z = z(x,C,D)$$

called characteristic curves. If we solve for C,D we get

$$C = \phi(x,y,z), \quad D = \psi(x,y,z).$$

Any solution of the given PDE is made up of characteristic curves determined by a condition of the form $F(C,D)=0$ and the general solution of the PDE in implicit form is

$$F(\phi(x,y,z),\psi(x,y,z)) = 0,$$

where F is any differentiable function of two variables.

Example 7. To solve the quasi-linear PDE

$$\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = 0$$

we have to solve the system

$$\frac{dy}{dx} = z,$$
$$\frac{dz}{dx} = 0.$$

The solutions of this system are $y = Dx + C$, $z = D$. If $z = u(x,y)$ is a solution then $z = D$ on the plane curve $y = Dx + C$ so that $D = f(C)$ so that the general solution is $z = f(y - xz)$. If we prescribe that $u(0,y) = -y$ we have $z = xz - y$ from which

$$z = \frac{y}{x-1}.$$

Note that the characteristic curve passing through the point

$$(0,-s,s)$$

of the curve $z = -y, x = 0$ on the surface $z = y/(x-1)$ has the equations

$$y = xs - s, \;\; z = s$$

which is a line passing through the point $(1, 0, s)$, a point on the line $x = 1, y = 0$. Every point on the surface $z = y/(x-1)$ is on exactly one of these lines an so the surface is a ruled surface.

Example 8. The characteristics curves of the quasi-linear PDE

$$\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = z$$

are the solutions of the system

$$\begin{aligned} \frac{dy}{dx} &= z, \\ \frac{dz}{dx} &= z. \end{aligned}$$

The general solution of the second equation is $z = Ce^x$. Putting this in the first equation gives

$$\frac{dy}{dx} = Ce^x$$

which has the solution $y = Ce^x + D$. If $z = u(x, y)$ is a solution, we have $y = Cx + D$ on the curve $z = Ce^x$ so that $D = f(C)$. Since $C = ze^{-x}$, $D = y - z$, the general solution is

$$ze^{-x} = f(y - z)$$

or $z = e^x f(y - z)$. If we prescribe $u(0, y) = -y$ then $-y = f(2y)$ or $f(y) = -y/2$ from which $z = e^x(z - y)/2$ which gives

$$z = \frac{ye^x}{e^x - 2}.$$