189-319B: Partial Differential Equations (April 2001)

Do ALL questions
All questions are of equal value
Calculators are allowed

1. (a) Using the method of characteristics, solve the PDE

$$\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = f(x),$$

where $f(x)$ is a given continuous function on the real line.

(b) Using (a), find the general solution of the PDE

$$\frac{\partial^2 u}{\partial x^2} - c^2\frac{\partial^2 u}{\partial y^2} = 0.$$

2. Solve the initial value problem

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 8\frac{\partial u}{\partial y} - 3u = 0,$$

$$u(0, y) = e^y, \quad \frac{\partial u}{\partial x}(0, y) = 0.$$

3. (a) Find the Fourier cosine series for the function $f(x) = x^2, \quad 0 \leq x \leq 1$. For what values of x does this series converge? Graph the function $F(x)$ defined by this series.

(b) If $F(x)$ is the function in (a), when does $F'(x)$ exist. Graph this this function. What is its Fourier series and when does this series converge to $F(x)$?

4. Find the temperature at any time $t \geq 0$ of a thin rod of unit length, insulated along its length and at both ends and which has unit mass per unit length and unit conductivity, if the initial temperature x units from one end of the rod is $f(x) = 2 + 3\cos(2\pi x)$.

5. Find a continous function $u(x, y)$ on the unit disk $x^2 + y^2 \leq 1$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the interior of the disk and such that

$$u(\cos(\theta), \sin(\theta)) = 1 + 2\sin(\theta) + 3\cos(2\theta)$$

for $0 \le \theta \le 2\pi$. Using Green's Theorem, show that this solution is unique.

6. Solve the initial value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad x^2 + y^2 < 1,\ t > 0$$

$$u(\cos(\theta), \sin(\theta), t) = 0, \quad 0 \le \theta \le 2\pi,\ t \ge 0$$

$$u(x, y, 0) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = f(r), \quad r = \sqrt{x^2 + y^2} \le 1$$

What is a physical problem described by these equations?