McGill University
Math 270A: Applied Linear Algebra
Test 2: Thursday November 18, 1999

Attempt All Questions
Justify All Your Assertions

1. Let W be the solution space of the system of equations

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 0 \end{aligned}$$

and let $T : W \to \mathbb{R}^2$ be defined by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_1 - x_2)$. Show that T is an isomorphism of vector spaces.

2. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $T(X) = X^T - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} X$.

   (a) Show that T is linear.

   (b) Find bases for the kernel and image of T.

   (c) Find the matrix of T with respect to the basis

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

   (a) Show that the characteristic polynomial of A is $(\lambda - 2)^2(\lambda - 5)$.

   (b) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.