McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Test 2

1. To show T is an isomorphism, we have to show it is linear, one-to-one and onto.

   (a) T linear: Let $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \in W$. Then $T(ax + by) = (ax_1 + by_1 + ax_2 + by_2, ax_1 + by_1 - ax_2 - by_2) = aT(x) + bT(y)$.

   (b) T one-to-one: Since T is linear, it suffices to show that $\text{Ker}(T) = \{0\}$. Now, $T(x) = 0$ implies $x_1 + x_2 = x_1 - x_2 = 0$ which implies $x_1 = x_2 = 0$. But since $x \in W$ we have $2x_3 + x_4 = x_3 + 2x_4 = 0$ which implies $x_3 = x_4 = 0$ and $x = 0$.

   (c) T onto: W is 2-dimensional with basis $u = (-2, 1, 0, 0), v = (-1, 0, 0, 1)$ and $T(u) = (-1, -3), T(v) = (-1, -1)$. Since $T(u), T(v)$ are linearly independent vectors in the 2-dimensional space $\mathbb{R}^2$ they span $\mathbb{R}^2$ and hence the $\text{Im}(T) = \mathbb{R}^2$.

2. (a) $T(aX + bY) = (aX + bY)^T + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(aX + bY) = aX^T + bY^T + a\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}X + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}Y = aT(X) + bT(Y)$.

   (b) If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $T(X) = \begin{bmatrix} a-c & c-d \\ b-a & d-b \end{bmatrix}$ so that $T(X) = 0 \iff a = b = c = d \iff X = a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Hence $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is a basis for $\text{Ker}(T)$. The image of T is generated by

$$T(E_1) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},\ T(E_2) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},\ T(E_3) = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},\ T(E_4) = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

   These vectors are linearly dependent since their sum is zero. Since the first three are linearly independent the first three are a basis for $\text{Im}(T)$.

   (c) By (b), $T(E_1) = E_1 - E_3,\ T(E_2) = E_3 - E_4,\ T(E_3) = -E_1 + E_2,\ T(E_4) = -E_2 + E_4$ so that the matrix of T with respect to the basis E_1, E_2, E_3, E_4 is

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

3. (a) $\begin{vmatrix} \lambda-3 & -1 & -1 \\ -1 & \lambda-3 & -1 \\ -1 & -1 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda-3 & 0 & -1 \\ -1 & \lambda-2 & -1 \\ -1 & 2-\lambda & \lambda-3 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-3 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda-3 & 0 & -1 \\ -1 & 1 & -1 \\ -2 & 0 & \lambda-4 \end{vmatrix}$
   $= (\lambda-2)\begin{vmatrix} \lambda-3 & -1 \\ -2 & \lambda-4 \end{vmatrix} = (\lambda-2)(\lambda^2 - 5\lambda + 5) = (\lambda-2)^2(\lambda-5)$.

   (b) The eigenspace of A for the eigenvalue 2 is the null space of the matrix $\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$. Solving $x_1 + x_2 + x_3 = 0$, we get $[1, -1, 0]^T, [1, 0, -1]^T$ as a basis for the eigenspace of A for the eigenvalue 2. The eigenspace of A for the eigenvalue 5 is the null space of the matrix $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. Solving $2x_1 - x_2 - x_3 = 0, -x_1 + 2x_2 - x_3 = 0, -x_1 - x_2 + 2x_3 = 0$ we get $[1, 1, 1]^T$ as a basis for the eigenspace of A for eigenvalue 5. If

$$P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix},$$

   we have $B = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ since B is the matrix of $T_A(X) = AX$ with respect to the basis formed by the colums of P which are eigenvectors of T_A with eigenvalues $2, 2, 5$ respectively.