McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Test 1

1. (a) The zero vector of V is the zero function 0. Since $0(0) = 0(1) = 0$, we have $0 \in W$. If $f, g \in W$ and $a, b \in \mathbb{R}$, we have $(af + bg)(0) = af(0) + bg(0) = 0$, $(af + bg)(1) = af(1) + bg(1) = 0$ which implies $af + bg \in W$. Hence W is a subspace of V.

   (b) If $f(x) = x, g(x) = 1 - x$ then $f(0) = 0, g(1) = 0$ which imply $f, g \in W$. But $(f + g)(x) = 1$ which implies that $f + g \notin W$.

2. (a) We have $x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = 0$ if and only if

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \\ -x_1 + x_2 - 3x_3 - 2x_4 &= 0 \\ x_1 - 2x_2 + 4x_4 + 3x_3 &= 0 \end{aligned}$$

   which has as solutions $(x_1, x_2, x_3, x_4) = a(2, -1, -1, 0)$. Thus, up to a scalar multiple, the only dependence relation is $2A_1 - A_2 - A_3 = 0$. From this is follows that $A_3 \in \text{span}(A_1, A_2)$ and A_1, A_2, A_4 are linearly independent and span W. Thus $\dim(W) = 3 < \dim(\mathbb{R}^{2\times 2} = 4$ which implies $W \neq \mathbb{R}^{2\times 2}$.

   (b) In (a) we found that A_1, A_2, A_4 was a basis of W.

3. (a) Since elmentary row operations don't change the row space of a matrix, and since the non-zero rows of a matrix in echelon form are a basis for the row space of this matrix, we see that $[1, 0, 2, 0, 1], [0, 1, 1, 1]$ is a basis for the row space of A. Moreover, since the pivot columns of A are a basis for the column space of A we see that $[7, 8, 6, 2]^T, [8, 7, 9, -2]^T$ is a basis for the column space of A. Since the null space of $A = \{X \mid AX = 0\}$ and $AX = 0 \iff X = a[2, -1, 1, 0, 0]^T + b[0, -1, 0, 1, 0]^T + c[-1, -1, 0, 0, 1]^T$, the vectors $[-2, -1, 1, 0, 0]^T, [0, -1, 0, 1, 0]^T, [-1, -1, 0, 0, 1]^T$ span the null space and are a basis since they are linearly independent.

   (b) The rank of a matrix is the dimension of the row space and the nullity of a matrix is the dimension of its null space. Hence, by (a), we have $\text{rank}(A) = \text{rank}(A^T) = 2$. Since rank + nullity = # of cols, we have $\text{nullity}(A) = 5 - 2 = 3$, $\text{nullity}(A^T) = 4 - 2 = 2$.

4. (a) We have $u, v \in W$ since $5u_{n+1} - 6u_n = 5 \cdot 2^{n+1} - 6 \cdot 2^n = (10 - 6)2^n = 2^{n+2} = u_{n+2}$ and $5v_{n+1} - 6v_n = 5 \cdot 3^{n+1} - 6 \cdot 3^n = (15 - 6)3^n = 3^{n+2} = v_{n+2}$. Now $au + bv = 0$ implies $a + b = 0, 2a + 3b = 0$ which implies $a = b = 0$ and hence that u, v are linearly independent. Since $\dim(W) = 2$ the vectors also span W since any n linearly independent vectors in an n-dimensional vector space V are a basis for V.

   (b) If $w \in W$ we have $w = au + bv$ by (a). But then $a + b = 1, 2a + 3b = 1$ which has the only solution $a = 2, b = -1$. Hence the coordinate vector of w with respect to the basis u, v is $[2, -1]^T$.