McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Assignment 5

1. (a) By the solution sheet for test 2, $[1,-1,0]^T, [1,0,-1]^T$ is a basis for the eigenspace for the eigenvalue 2 and $[1,1,1]^T$ is a basis for the eigenspace for the eigenvalue 5. Applying the Gram-Schmidt process to the first two eigenvectors we get $[1,-1,0]^T, [1,1,-2]^T$. Then $[1,-1,0]^T, [1,1,-2]^T, [1,1,1]^T$ is an orthogonal basis of $\mathbb{R}^3$ consisting of eigenvectors of A. Normalizing these vectors we get the vectors

$$u_1 = [1/\sqrt{2}, -1/\sqrt{2}, 0]^T,\ u_2 = [1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}]^T,\ u_3 = [1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]^T.$$

If P is the matrix whose columns are respectively u_1, u_2, u_3, we have

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

(b) If

$$P_1 = u_1u_1^T = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad P_2 = u_2u_2^T = \begin{bmatrix} 1/6 & 1/6 & -1/3 \\ 1/6 & 1/6 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix},\quad P_3 = u_3u_3^T = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix},$$

we have $A = 2P_1 + 2P_2 + 5P_3,\ P_i^2 = P_1,\ P_iP_j = 0\ (i \neq j),\ I = P_1 + P_2 + P_3$. The required matrices Q_1, Q_2 are

$$Q_1 = P_1 + P_2 = \begin{bmatrix} 4/3 & -1/3 & -1/3 \\ -1/3 & 4/3 & -2/3 \\ -1/3 & -1/3 & 4/3 \end{bmatrix},\quad Q_2 = P_3 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

(c) The required matrix is

$$B = \sqrt{2}Q_1 + \sqrt{5}Q_3 = \begin{bmatrix} (4\sqrt{2}+\sqrt{5})/3 & (\sqrt{5}-\sqrt{2})/3 & (\sqrt{5}-\sqrt{2})/3 \\ (\sqrt{5}-\sqrt{2})/3 & (4\sqrt{2}+\sqrt{5})/3 & (\sqrt{5}-\sqrt{2})/3 \\ (\sqrt{5}-\sqrt{2})/3 & (\sqrt{5}-\sqrt{2})/3 & (4\sqrt{2}+\sqrt{5})/3 \end{bmatrix}.$$

(d) If we make the change of variable $[x,y,z]^T = P[u,v,w]^T$, where P is as in question 1, we have $Q(x,y,z) = 2u^2 + 2v^2 + 5ww^2$ and $u^2+v^2+w^2 = x^2+y^2+z^2$. If $x^2+y^2+z^2 = 1$, we have

$$2 \leq 2(u^2+v^2+w^2) \leq Q(x,y,z) \leq 5(u^2+v^2+w^2) = 5$$

so that the minimum and maximum values of $Q(x,y,z)$ on the unit sphere are respectively 2 and 5.

(e) Making the above change of variable, the integral becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2u^u-2v_2-5w^2}dudvdw = (\int_{-\infty}^{\infty} e^{-2u^2}du)(\int_{-\infty}^{\infty} e^{-2v^2}dv)(\int_{-\infty}^{\infty} e^{-5w^2}dw) = (\sqrt{\pi}/\sqrt{2})^2(\sqrt{\pi}/\sqrt{5}) = \pi^{3/2}/2\sqrt{5}$$

using the fact that $\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\pi}/\sqrt{a}$.

2. We have $A^* = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ and $AA^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = A^*A$ which implies that A is normal. The vectors $[1,1]^T,\ [1,-1]^T$ are eigenvectors of A with eigenvalues $1+i, 1-i$ respectively. The required matrix U is therefore $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.

3. The matrix $A^TA = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ has eigenvectors $[1,1]^T, [1,-1]^T$ with corresponding eigenvalues 1 and 3. Then $u_1 = [1/\sqrt{2}, -1\sqrt{2}]^T,\ u_2 = [1/\sqrt{2}, 1/\sqrt{2}]^T$ is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A^TA. Let $v_1 = Au_1/\sqrt{3} = [2/\sqrt{6}, -1/\sqrt{6}, 1/\sqrt{6}]^T,\ v_2 = Au_1 = [0, 1/\sqrt{2}, 1/\sqrt{2}]^T$. We complete the orthogonal unit vectors v_1, v_2 to an orthonormal basis of $\mathbb{R}^3$ by adding $v_3 = [1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}]^T$. The singular value decomposition of A is

$$A = QDP^T = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

The least squares solution X of $AX = b$ with $b = [1,1,1]^T$ minimizes

$$||AX - b|| = ||QDP^TX - b|| = ||DY - Q^Tb|| = ((\sqrt{3}y_1 - 2/\sqrt{6})^2 + (y_2 - \sqrt{2})^2 + 1/3)^{1/2}$$

where $[y_1, y_2]^T = Y = P^TX$. The minimum is $1/\sqrt{3}$ and occurs when $y_1 = 2/3\sqrt{2},\ y_2 = \sqrt{2}$. Since $X = PY$, the least squares solution is $x_1 = 4/3,\ x_2 = 2/3$.