McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Assignment 5

1. The system can be written as $X_{n+1} = AX_n$, where $X_n = [x_n, y_n]^T$ and $A = \begin{bmatrix} .6 & .5 \\ -.1 & 1.2 \end{bmatrix}$. Its general solution is $X_n = A^n X_0$. The matrix $A$ has $u = [1,1]^T, v = [5,1]^T$ as eigenvectors with eigenvalues $1.1, .7$ respectively and we have $X_0 = au + bv$ with $a = (5y_0 - x_0)/4, b = (x_0 - y_0)/4$. Hence $X_n = A^n X_0 = aA^n u + bA^n v = a(1.1)^n u + b(.7)^n v$ so that $x_n = (1.1)^n a + (.7)^n b$, $y_n = (1.1)^n a + 5(.7)^n b$ is the general solution. Since $x_n - y_n = -4(.7)^n b$, we see that $x_n - y_n$ tends to zero as $n$ tends to infinity. If $x_0 = 10, y_0 = 30$ we have $x_n = 35(1.1)^n - 25(.7)^n,\ y_n = 35(1.1)^n - 5(.7)^n$

2. The solution set of the linear recurrence equation is the kernel of $L^2 - 1.8L + .77 = (L - 1.1)(L - .7)$. Thus

$$\mathrm{Ker}(L^2 - 1.8L + .77) = \mathrm{Ker}(L - 1.1) + \mathrm{Ker}(L - .7) = \mathrm{span}((1, 1.1, \dots, (1.1)^n, \dots)) + \mathrm{span}((1, .7, \dots, (.7)^n, \dots))$$

so that the general solution of the given recurrence equation is $x_n = a(1.1)^n + b(.7)^n b$. Setting $n = 0, 1$ we have $x_0 = a + b, x_1 = 1.1a + .7b$, so that $a = (10x_1 - 7x_0)/4,\ b = (11x_0 - 10x_1)/4$. In particular, if $x_0 = 1, x_1 = 2$, we have $x_n = (13/4)(1.1)^n - (9/4)(.7)^n$ while, if $x_0 = 2, x_1 = 1$, we have $x_n = -(1.1)^n + 3(.7)^n$.

3. The solution space of this equation is the kernel of $D^4 - 10D^3 + 37D^2 - 60D + 36 = (D-2)^2(D-3)^2$. Thus

$$\mathrm{Ker}(D^4 - 10D^3 + 37D^2 - 60D + 36) = \mathrm{Ker}((D-2)^2) + \mathrm{Ker}((D-3)^2) = \mathrm{span}(e^{2x}, xe^{2x}) + \mathrm{span}(e^{3x}, xe^{3x}).$$

Hence the general solution of the given differential equation is $f(x) = ae^{2x} + bxe^{2x} + ce^{3x} + dxe^{3x}$. Using the given ititial conditons, we find that $a, b, c, d$ are satisfy

$$\begin{aligned} a + c &= 1, \\ 2a + b + 3c + d &= 1, \\ 4a + 4b + 9c + 6d &= 1, \\ 8a + 12b + 27c + 27d &= 1 \end{aligned}$$

which has for unique solution $a = -4, b = -4, c = 5, d = -2$ so that $f(x) = -4e^{2x} - 4xe^{2x} + 5e^{3x} - 2xe^{3x}$.

4. The general solution of $\frac{dX}{dt} = AX$ is $X = e^{At}X(0)$. The column vectors $u = [1,-1,0]^T, v = [1,0,-1]^T, w = [1,1,1]^T$ are eigenvectors of $A$ with eigenvalues $-4, -4, -1$ respectively. Since

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1},$$

$$\text{we have } e^{At} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-4t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1}.$$

$$\text{Since } \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & +1/3 \end{bmatrix},$$

$$e^{At} = \begin{bmatrix} (2/3)e^{-4x} + (1/3)e^{-x} & -(1/3)e^{-4x} + (1/3)e^{-x} & -(1/3)e^{-4x} + (1/3)e^{-x} \\ -(1/3)e^{-4x} + (1/3)e^{-x} & (2/3)e^{-4x} + (1/3)e^{-x} & -(1/3)e^{-4x} + (1/3)e^{-x} \\ -(1/3)e^{-4x} + (1/3)e^{-x} & (2/3)e^{-4x} + (1/3)e^{-x} & (-1/2)e^{-4x} + (3/2)e^{-x} \end{bmatrix}.$$

It follows that

$$\begin{aligned} x(t) &= ((2/3)x(0) - (1/3)y(0) - (1/3)z(0))e^{-4x} + ((1/3)x(0) + (1/3)y(0) + (1/3)z(0))e^{-x} \\ y(t) &= (-(1/3)x(0) + (2/3)y(0) - (1/3)z(0))e^{-4x} + ((1/3)x(0) + (1/3)y(0) + (1/3)z(0))e^{-x} \\ z(0) &= (-(1/3)x(0) - (1/3)y(0) + (2/3)z(0))e^{-4x} + ((1/3)x(0) + (1/3)y(0) + (1/3)z(0))e^{-x} \end{aligned}$$

from which it follows that $x(t), y(t), z(t)$ converge to zero as $t$ tends to infinity. If $x(0) = 1, y(0) = 2, z(0) = 3$ we have $x(t) = -e^{-4x} + 2e^{-x},\ y(t) = 2e^{-x},\ z(t) = e^{-4x} + 2e^{-x}$.