McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Assignment 4

1. Apply the Gram-Schmidt process to the column vectors $v_1 = [1,0,1]^T, v_2 = [2,1,4], v_3 = [4,1,6]^T$ to get $u_1 = v_1, u_2 = v_2 - (3/2)u_1 = [-1,1,1]^T = 0, v_3 = u_3 - (5/2)u_1 - (3/8)u_2$. Then $u_1 = v_1, u_2 = 3v_1 + v_2, u_3 = 5v_1 + v_2$ which gives

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \end{bmatrix}.$$

Dividing each column of the first matrix in the above product by its length and mulitiplying the corresponding row of the second matrix by this number, we get the QR-decomposition $A = QR$ of A:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 4 \\ 1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 5/\sqrt{2} \\ 0 & \sqrt{3} & \sqrt{3} \end{bmatrix}.$$

The least squares solution of $AX = Y$ are the solutions of $QRX = QQ^TY$. Since $Q^TQ = I$, the least squares solutions of $AX = Y$ are the solutions of $RX = Q^TY$. If $Y = [1,1,1]^T$ and $X = [x,y,z]^T$x, these equations are $\sqrt{2}x + 3\sqrt{2}y + 5\sqrt{2} = \sqrt{2}$, $\sqrt{3}y + \sqrt{3}z - \sqrt{3}/3$, i.e., $x + 3y + 5z = 1, y + z = 1/3$ which has for solutions $x = -2t, y = -t + 1/3, z = t$ with t arbitrary.

2. (a) $T(aX+bY) = C(aX+bY)-(aX+bY)C = aCX+bCY-aXC-bYC = a(CX-XC) = b(CY-YC) = aT(X)+bT(Y)$.

   (b) Since $T(I) = T(C) = 0$ and I, C are linearly independent, we have $\dim \mathrm{Ker}(T) \geq 2$.

   (c) $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} 2(c-b) & 2(d-a) \\ 2(a-d) & 2(b-c) \end{bmatrix} = 2(c-b)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2(d-a)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which shows that $\mathrm{Ker}(T) = \mathrm{span}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$; since these two matrices are linearly indendent they form a basis for the kernel of T. Also $\mathrm{Im}(T) = \mathrm{span}(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$ since $2(c-b) = \alpha, 2(d-a) = \beta$ is solvable for any α, β; since these two matrices are linearly independent they form a basis for the mage of T.

   (d) $T^3(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = T^2(\begin{bmatrix} 2(c-b) & 2(d-a) \\ 2(a-d) & 2(b-c) \end{bmatrix}) = T(\begin{bmatrix} 8(a-d) & 8(b-c) \\ 8(c-b) & 8(d-a) \end{bmatrix} = \begin{bmatrix} 32(c-b) & 32(d-a) \\ 32(a-d) & 32(b-c) \end{bmatrix} = 16T(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$.

3. (a) $T((ap(t) + bq(t)) = t^2(ap(t) + bq(t))'' - 2(ap(t) + bq(t)) = t^2(ap''(t) + bq''(t)) - 2ap(t) - 2bq(t) = a(t^2p''(t) - 2p(t)) + b(t^2q''(t) - 2q(t)) = aT(p(t)) + bT(q(t))$.

   (b) $T(p(t)) = t^2(2a_2 + 6a_3t) - 2(a_0 + a_1t + a_2t^2 + a_3t^3) = -2a_0 - 2a_1t + 4a_3t^3$. Hence $T(p(t)) = 0 \iff a_0 = a_1 = a_3 = 0$ which shows that the polynomial t^2 is a basis for the kernal of T. We also see that $1, t, t^3$ is a basis for the image of T.

   (c) $T(1) = -2, T(t) = -2t, T(t^2) = 0, T(t^3) = 4t^3$ implies that the matrix of T with respect to the basis $1, t, t^2, t^3$ is

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

$T(1+t) = -2-2t = -2(1+t), T(1-t) = -2+2t = -2(1-t), T(t^2+t^3) = 4t^3 = 2(t^2+t^3) - 2(t^2-t^3), T(t^2-t^3) = -4t^3 = -2(t^2+t^3) + 2(t^2-t^3)$ implies that the matrix of T with respect to the basis $1+t, 1-t, t^2+t^3, t^2-t^3$ is

$$B = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}.$$

The transition matrix from the first basis to the second is

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

With this P we have $B = P^{-1}AP$.

4. (a) Since $T(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}) = \begin{bmatrix}0 & -2\\2 & 0\end{bmatrix}, T(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} = \begin{bmatrix}-2 & 0\\0 & 2\end{bmatrix}, T(\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}) = \begin{bmatrix}2 & 0\\0 & -2\end{bmatrix}, T(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}$

$$A = \begin{bmatrix}0 & -2 & 2 & 0\\-2 & 0 & 0 & 2\\2 & 0 & 0 & -2\\0 & 2 & -2 & 0\end{bmatrix}.$$

The characteristic polynomial of T is

$$\det(\lambda I - A) = \begin{vmatrix}\lambda & 2 & -2 & 0\\2 & \lambda & 0 & -2\\-2 & 0 & \lambda & 2\\0 & -2 & 2 & \lambda\end{vmatrix} = \lambda^4 - 16\lambda^2 = \lambda^2(\lambda-4)(\lambda+4)$$

which shows that the eigenvalues of A or T are $0, 4, -4$.

(b) The eigenspace of A for the eigenvalue 0 is the null space of the matrix

$$A = \begin{bmatrix}0 & -2 & 2 & 0\\-2 & 0 & 0 & 2\\2 & 0 & 0 & -2\\0 & 2 & -2 & 0\end{bmatrix}.$$

A basis for the null space of this matrix is $[1,0,0,1]^T, [0,1,1,0]^T$. The eigenspace for the eigenvalue 4 is the null space of the matrix

$$4I - A = \begin{bmatrix}4 & 2 & -2 & 0\\2 & 4 & 0 & -2\\-2 & 0 & 4 & 2\\0 & -2 & 2 & 4\end{bmatrix}.$$

A basis for the null space of this matrix is $[1,-1,1,-1]^T$. The eigenspace for the eigenvalue -4 is the null space of the matrix

$$-4I - A = \begin{bmatrix}-4 & 2 & -2 & 0\\2 & -4 & 0 & -2\\-2 & 0 & -4 & 2\\0 & -2 & 2 & -4\end{bmatrix}.$$

A basis for the nullspace of this matrix is $[1,1,-1,-1]^T$.

(c) The eigenvectors $u_1 = [1,0,0,1]^T, u_2 = [0,1,1,0]^T, u_3 = [1,-1,1,-1]^T, u_4 = [1,1,-1,-1]^T$ are linearly independent and hence a basis of $\mathbb{R}^{4\times 1}$. The matrix P whose columns are u_1, u_2, u_3, u_4 is therefore invertible and

$$P^{-1}AP = \begin{bmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 4 & 0\\0 & 0 & 0 & -4\end{bmatrix}$$

since $Au_1 = 0, Au_2 = 0, Au_3 = 4u_3, Au_4 = -4u_4$.