McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Assignment 3

1. Define $T : V_{\mathbb{C}} \to \mathbb{C}^X$ by $T(f_1, f_2) = f_1 + if_2$. Then T is one-to-one and onto since every $f \in \mathbb{C}^X$ can be uniquely written in the form $f = f_1+if_2$ with $f_1, f_2 \in \mathbb{R}^X$. Now $T((f_1, f_2)+(g_1, g_2)) = T((f_1+g_1, f_2+g_2) = ((f_1+g_1)+i(f_2+g_2) = f_1+ig_1+f_2+if_2 = T(f_1, f_2)+T(g_1, g_2)$ and $T((a+bi)(f, g)) = T(af-bg, ag+bf) = (af-bg)+i(ag+bf) = (a+bi)(f+ig) = (a+bi)T(f, g)$ which shows that T is also linear.

2. (a) If $u = (a_1, b_1, c_1, d_1, e_1, f_1), v = (a_2, b_2, c_2, d_2, e_2, f_2) \in W_S$ and $\alpha, \beta \in \mathbb{R}$ then, for all $(x, y) \in S$, we have $a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1 = 0, a_2x^2 + b_2xy + c_2y^2 + d_2x + e_2y + f_2 = 0$ and so $\alpha(a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1) + \beta(a_2x^2 + b_2xy + c_2y^2 + d_2x + e_2y + f_2) = 0$ which implies that
$$(\alpha a_1 + \beta a_2)x^2 + (\alpha b_1 + \beta b_2)xy + (\alpha c_1 + \beta c_2)y^2 + (\alpha d_1 + \beta d_2)x + (\alpha e_1 + \beta e_2)y + \alpha f_1 + \beta f_2 = 0$$
and hence that $\alpha u + \beta v \in W_S$.

   (b) In this case W_S is the solution space of the system
$$\begin{aligned} a + b + c + d + e + f &= 0 \\ a + 10b + 25c + 2a + 5e + f &= 0 \\ 9a + 3d + f &= 0 \\ 16a + 24b + 36c + 4d + 6e + f &= 0 \end{aligned}$$
and so $\dim W_S = 6 - \operatorname{rank}(A)$ where
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 10 & 25 & 2 & 5 & 1 \\ 9 & 0 & 0 & 3 & 0 & 1 \\ 16 & 24 & 36 & 4 & 6 & 1 \end{bmatrix}.$$
So $\dim W_S = 2$ if an only if the column rank of A is 4. But, as is easily checked, columns $2, 3, 5, 6$ are linearly independent.

   (c) The degenerate conics $(4x - y - 3)(6x - y - 18) = 0$, $(x + 2y - 3)(x - 2y + 8) = 0$ pass through the four given points. Expanding, we find that the conics $24x^2 - 10xy + y^2 - 90x + 21y + 54 = 0$, $x^2 - 4y^2 + 5x + 24y - 24 = 0$ pass through the given four points and so $u = (24, -10, 1, -90, 21, 54), v = (1, 0, -4, 5, 24, -24) \in W_S$. Since u, v are linearly independent and $\dim W_S = 2$, they are a basis for W_S.

   (d) From (c) it follows that the general equation of a conic which passes through S is $\alpha(24x^2 - 10xy + y^2 - 90x + 21y + 54) + \beta(x^2 - 4y^2 + 5x + 22y - 24) = 0$. Such a conic also passes through $(-1, 1)$ if and only if $200\alpha - 10\beta = 0$, i.e., $\beta = 20\alpha$. hence the conic which passes through S and $(-1, 1)$ has the equation $44x^2 - 10xy - 79y^2 + 10x + 461y - 426 = 0$.

3. (a) Since $T(a, b, c) = 0$ implies $a + bx + cx^2 = 0$ for all $0 \le x \le 1$ we must have $a = b = c = 0$ since a non-zero polynomial of degree ≤ 2 has at most 2 roots. Also, T is onto, since $f \in W$ implies there are $a, b, c \in \mathbb{R}$ with $f(x) = a + bx + cx^2$ for $0 \le x \le 1$ from which $f = T(a, b, c)$. Finally, T is linear since $T(\alpha(a_1, b_1, c_1) + \beta(a_2, b_2, c_2)) = T(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1 + \beta c_2) = g$ where $g(x) = \alpha a_1 + \beta a_2 + (\alpha b_1 + \beta b_2)x + (\alpha c_1 + \beta c_2)x^2 = \alpha(a_1 + b_1x + c_1x^2) + \beta(a_2 + b_2x + c_2x^2)$ which implies that $g = \alpha T(a_1, b_1, c_1) + \beta T(a_2, b_2, c_2)$.

   (b) We have (1) $< u, u > = < T(u), T(u) > \ge 0$ with equality $\iff T(u) = 0 \iff u = 0$ since T is $1-1$; (2) $< u, v > = < T(u), T(v) > = < T(v), T(u) > = < v, u >$, (3) $< au + bv, w > = < T(au + bv), T(w) > = < aT(u) + bT(v), T(w) > = a < T(u), T(v) > + b < T(v), T(w) > = a < u, w > + b < v, w >$. Thus $< u, v > = < T(u), T(v) >$ is an inner product on V. Now
$$||(a, b, c)||^2 = \int_0^1 (a + bx + cx^2)^2 dx = a^2 + ab + b^2/3 + ac + 2bc/3 + c^2/5.$$

4. Since the vectors u_1, u_2, u_3 are othogonal and non-zero, the orthogonal project of $v = (a, b, c, d)$ on W is
$$\begin{aligned} w &= \frac{< v, u_1 >}{< u_1, u_1 >}u_1 + \frac{< v, u_2 >}{< u_2, u_2 >}u_2 + \frac{< v, u_3 >}{< u_3, u_3 >}u_3 \\ &= \frac{a + 2b + 2c + d}{10}(1, 2, 2, 1) + \frac{a + b - c - d}{4}(1, 1, -1, -1) + \frac{2a - b - c + 2d}{10}(2, -1, -1, 2) \\ &= (1/4)(3a + b - c + d, a + 3b + c - d, -a + b + 3c + d, a - b + c + 3d). \end{aligned}$$
The vector (x, y, z) is a least squares solution of the given system if and only if $xu_1 + yu_2 + zu_3 = w$. Since u_1, u_2, u_3 are orthogonal and non-zero, this yields
$$x = \frac{< w, u_1 >}{< u_1, u_1 >} = (a + 2b + 2c + d)/10,\ y = \frac{< w, u_2 >}{< u_2, u_2 >} = (a + b - c - d)/4,\ z = \frac{< w, u_3 >}{< u_3, u_3 >} = (2a - b - c + 2d)/10.$$

5. Ww first find an orthogonal basis for W by applying the Gram Schmidt process to the basis $1, x, x^2$ of W. This yields the polynomials

$$h_1(x) = 1,\ h_2(x) = x - \frac{< x, h_1 >}{< h_1, h_1 >} h_1 = x - 1/2,\ , h_3(x) = x^2 - \frac{< x^2, h_2 >}{< h_2, h_2 >} h_2 - \frac{< x^2, h_1 >}{< h_1, h_1 >} h_1 = x^2 - (x - 1/2) - 1/3 = x^2 - x - 1/6.$$

Now the orthogonal projection of $f(x) = e^x$ on W is

$$\begin{aligned} g(x) &= \frac{< f, h_1 >}{< h_1, h_1 >} h_1(x) + \frac{< f, h_2 >}{h_2, h_2 >} h_2(x) + \frac{< f, h_3 >}{< h_3, h_3 >} h_3(x) \\ &= e - 1 + (18 - 9e)(x - 1/2) + (210e - 570)(x^2 - x + 1/6) \\ &= 39e - 105 + (588 - 216e)x + (210e - 570)x^2. \end{aligned}$$

The function g is the best approximation to $f(x) = e^x$ by a function $h(x) = a + bx + cx^2$ in the sense that

$$\int_0^1 (e^x - a - bx - cx^2)^2 dx \ \geq \int_0^1 (e^x - g(x))^2 dx$$

with equality if an only if $a + bx + cx^2 = g(x)$.