McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Assignment 2

1. We have to verify that the vector space axioms hold.
   1. Assoc. Law: $(v_1, w_1) + ((v_2, w_2) + (v_3, w_3)) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = (v_1 + (v_2 + v_3), w_1 + (w_2, w_3)) = ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3)) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = ((v_1, w_1) + (v_2, w_2)) + (v_3, w_3)$.
   2. Comm. Law: $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$.
   3. Existence of Zero: $(0, 0) + (v, w) = (0 + v, 0 + w) = (v, w) \implies (0, 0)$ is the zero vector.
   4. Existencs of Additive Inverses: $(-v, -w) + (v, w) = (-v + v, -w + w) \implies (-v, -w)$ is the additive inverse of $(v, w)$.
   5. Assoc. of Scalar Mult.:$a(b(v, w)) = a(bv, bw) = ((a(bv), a(bw)) = ((ab)v, (ab)w) = (ab)(v, w)$.
   6. Mult. by 1: $1(v, w) = (1v, 1w) = (v, w)$.
   7. Dist. Laws: $(a + b)(v, w) = ((a + b)v, (a + b)w) = (av + bv, aw + bw) = (av, aw) + (bv, bw) = a(v, w) + b(v, w)$; $a((v_1, w_1) + (v_2, w_2)) = a(v_1 + v_2, w_2 + w_2) = (a(v_1 + v_2), a(w_1 + w_2)) = (av_1 + av_2, aw_1 + aw_2) = (av_1, aw_1) + (av_2, aw_2) = a(v_1, w_1) + a(v_2, w_2)$.

2. Using problem 1 in the case $W = V$, we only have to check axioms 5 and 7.
   5. $(a + bi)((c + di)(u, v)) = (a + bi)(cu - dv, cv + du) = (a(cu - dv) - b(cv + du), a(cv + du) + b(cu - dv)) = (acu - adv - bcv - bdu, acv + adu + bcu - bdv) = ((ac - bd)u - (ad + bc)v, (ad + bc)u + (ac - bd)v) = ((ac - bd) + (ad + bc)i)(u, v) = ((a + bi)(c + di))(u, v)$.
   7. $((a + bi) + (c + di))(u, v) = ((a + c) + (b + d)i)(u, v) = ((a + c)u - (b + d)v, (a + c)v + (b + d)u) = (au + cu - bv - dv, av + cv + bu + du) = (au - bv, av + du) + (cu - dv, cv + du) = (a + bi)(u, v) + (c + di)(u, v)$;
   $(a + bi)((u_1, v_1) + (u_2, v_2) = (a + bi)(u_1 + u_2, v_1 + v_2) = (a(u_1 + u_2) - b(v_1 + v_2), a(v_1 + v_2) + b(u_1 + u_2) = (au_1 + au_2 - bv_1 - bv_2, av_1 + av_2 + bu_1 + bu_2) = (au_1 - bv_1, av_1 + bu_1) + (au_2 - bv_2, av_2 + bu_2) = (a + bi)(u_1, v_1) + (a + bi)(u_2, v_2)$
   Finally, $(u, v) = (u, 0) + (v, 0) = (u, 0) + i(v, 0)$.

3. (a) $W$ is the zero subspace as $x^2 + y^4 = 0 \implies x^2 = y^4 = 0 \implies x = y = 0$.
   (b) $W$ is not a subspace as $(i, 1), (-i, 1) \in W$ but $(i, 1) + (-i, 1) = (0, 2) \notin W$.
   (c) $W$ is a subspace as (a) the zero sequence $0 = (0, 0, \ldots, 0, \ldots) \in W$ since $0_n = 0$ for all $n$ implies $0_{n+2} = 0 = 50_{n+1} + 60_n$ for all $n$ and (b) $x, y \in W,\ a, b \in \mathbb{R} \implies (ax + by)_{n+2} = ax_{n+2} + by_{n+2} = a(5x_{n+1} + 6x_n) + b(5y_{n+1} + 6y_n) = 5(ax_{n+1} + by_{n+1}) + 6(ax_n + by_n)w = 5ax_{n+1} + 6ax_n + 5by_{n+1} + 6by_n = 5(ax_{n+1} + by_{n+1}) + 6(ax_n + by_n) = 5(ax + by)_{n+1} + 6(ax + by)_n$.
   (d) $W$ is a subspace since (a) the zero function is in $W$ as $0(x) = 0$ for all $x \implies 0(x) = 0 = 0(x^2)$ for all $x$ and (b) $f, g \in W,\ a, b \in \mathbb{R} \implies (a + bg)(x) = af(x) + bg(x) = af(x^2) + bg(x^2) = (af + bg)(x^2)$.
   (e) W is not a subspace as the constant function, $f(x) = 1$ for all $x$, is in $W$ but $2f$ is not.

4. (a) We have $au_1 + bu_2 + cu_3 + bu_4 \iff a + b + c + d = 0, 2a - b + 5c - 4d = 0, 3a + 2b + 4c + d = 0, 4a + 3b + 5c + 2d = 0$ which has the non-zero solution $a = 1, b = -2, c = 0, d = 1$
   (b) $v_1 = 1 + 2t + t^2, v_2 = 4 + 4t + t^2, v_3 = 9 + 6t + t^2, v_4 = 16 + 8t + t^2$ and so
   $$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = (a_1 + 4a_2 + 9a_3 + 16a_4) + (2a_1 + 4a_2 + 6a_3 + 8a_4)t + (a_1 + a_2 + a_3 + a_4)t^2$$
   which is zero iff
   $$a_1 + 4a_2 + 9a_3 + 16a_4 = 0, 2a_1 + 4a_2 + 6a_3 + 8a_4 = 0, a_1 + a_2 + a_3 + a_4 = 0$$
   a homogeneous system of 3 equations in 4 unknowns and so has a no-zero solution which implies that the given vectors are linearly dependent.
   (c) If $ae^x + be^{2x} + ce^{-x} = 0$ for all $x$ then, setting $x = 0$ we get $a + b + c = 0$. Differenting $ae^x + be^{2x} + ce^{-x} = 0$ twice and setting $x = 0$, we get $a + 2b - c = 0$ and $a + 4b + c = 0$ in addition to $a + b + c = 0$. The only solution of these three equations is $a = b = c = 0$ which implies that the given functions are linearly dependent.
   (d) These functions are linearly dependent as $-\sin^2(x) + \cos^2(x) - \cos(2x) = 0$ for all $x$.
   (e) $av_1 + bv_2 + cv_3 = 0 \iff a + (1 + i)b + c = 0,\ (2 + i)a + b = 0,\ ia + ib + ic = 0$ which has as only solution $a = b = c = 0$.

5. (a) The row reduced form of $A$ is $\begin{bmatrix} 1 & 4 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Since the row space of a matrix does not change under row reduction and the non-zero rows of a matrix in echelon form are a basis for its row space, we see that $(1, 4, 0, 2, 2), (0, 0, 1, 1, 0)$ is

a basis for the row space of $A$. Now $X^T = [x_1, x_2, x_3, x_4, x_5]$ is in the null space of $A \iff AX = 0 \iff x_1 + 4x_2 + 2x_4 + 2x_5 = 0,\ x_3 + x_4 = 0 \iff x_1 = -4x_2 - 2x_4 - 2x_5,\ x_3 = -x_4 \iff X^T = x_2[-4, 1, 0, 0, 0] + x_4[-2, 0, -1, 1, 0] + x_5[-2, 0, 0, 0, 1]$ with $x_2, x_4, x_5$ arbitrary. This show that $[-4, 1, 0, 0, 0]^T, [-2, 0, -1, 1, 0]^T, [-2, 0, 0, 0, 1]^T$ is a basis for the null space of $A$.

Since the pivot columns of $A$ form a basis for the column space of $A$, their transposes $(1, 2, 1, 1), (0, 1, 1, -1)$ form a basis for the row space of $A^T$. Then $[y_1, y_2, y_3, y_4]^T$ is in the null space of $A^T \iff y_1 + 2y_2 + y_3 + y_4 = 0,\ y_2 + y_3 - y_4 = 0 \iff [y_1, y_2, y_4] = y_3[1, -1, 1, 0] + y_4[-3, 1, 0, 1]$ which shows that $[1, -1, 1, 0]^T, [-3, 1, 0, 1]^T$ is a basis for the null space of $A^T$.

(b) First check that the f's and g's are in the solution space and then check the linear independence of the f' and of the g's. Since the solution space is also three dimensional, the f's and g's each form basis for it. Now $(2, -4, 1, -1, 8) = (29/15)f_1 + (23/15)f_2 + (2/3)f_3 = (1/6)g_1 + (34/3)g_2 + (-7/2)g_3$ which shows that the coordinate vector of $v$ is $(29/15, 23/15, 2/3)$ with respect to the $f$-basis and $(1/6, 34/3, -7/2)$ with respect to the $g$-basis.

(c) Since $g_1 = (2/15)f_1 + (29/15)f_2 + (-4/3)f_3, g_2 = (1/3)f_1 + (1/3)f_2 + (-1/3)f_3, g_3 = (8/15)f_1 + (11/15)f_2 + (-4/3)f_3$, the transition matrix from the $f$-basis to the $g$-basis is $P = \begin{bmatrix} 2/15 & 1/3 & 8/15 \\ 29/15 & 1/3 & 11/15 \\ -4/3 & -1/3 & -4/3 \end{bmatrix}$.

Since $f_1 = (-1/2)g_1 + 4g_2 + (-1/2)g_3, f_2 = (2/3)g_1 + (4/3)g_2 - g_3, f_3 = (1/6)g_1 + (7/3)g_2 + (-3/2)g_3$, the transition matrix from the $g$-basis to the $f$-basis is $Q = \begin{bmatrix} -1/2 & 2/3 & 1/6 \\ 4 & 4/3 & 7/3 \\ -1/2 & -1 & -3/2 \end{bmatrix}$. We have $PQ = QP = I$.