McGill University
Math 270A: Applied Linear Algebra
Solution Sheet for Assignment 1

1. (a) $\begin{bmatrix}1 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & -4\\ 7 & 3 & 9\end{bmatrix} = \begin{bmatrix}44 & 17 & 50\end{bmatrix}$ (b) $\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 4\\ 7 & 2 & 0\end{bmatrix} = \begin{bmatrix}23 & 5 & 4\\ 12 & 5 & -4\end{bmatrix}$

   (c) $\begin{bmatrix}4 & -1 & 2\\ 6 & 0 & 3\end{bmatrix}\begin{bmatrix}2\\ -6\\ 9\end{bmatrix} = \begin{bmatrix}32\\ 39\end{bmatrix}$ (d) $\begin{bmatrix}2 & 1\\ 4 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}-1 & 6 & 1 & 0 & 1\\ 3 & 2 & -1 & 2 & 0\end{bmatrix} = \begin{bmatrix}1 & 14 & 1 & 2 & 2\\ 5 & 30 & 1 & 6 & 4\\ 5 & 10 & -1 & 4 & 1\end{bmatrix}$

   (e) $\begin{bmatrix}2i & 1+i\\ -3 & i\end{bmatrix}\begin{bmatrix}6\\ 2+3i\end{bmatrix} = \begin{bmatrix}-1+17i\\ -21+2i\end{bmatrix}$ (f) $\begin{bmatrix}5\\ 3\end{bmatrix}\begin{bmatrix}2 & -6\end{bmatrix} = \begin{bmatrix}10 & -30\\ 6 & -18\end{bmatrix}$ (g) $\begin{bmatrix}2 & -6\end{bmatrix}\begin{bmatrix}5\\ 3\end{bmatrix} = \begin{bmatrix}-8\end{bmatrix}$

2. (a) If A is $p \times q$ then A^H is $q \times p$ so that $A^H = -A$ only if $p = q$, i.e, A is square. If $< A >_{ij} = a_{ij}$, then $< A^H >_{ij} = \overline{a}_{ij}$ so that $A^H = -A$ iff A is square and $\overline{a}_{ij} = -a_{ji}$ for all i, j. In particular, $\overline{a}_{ii} = -a_{ii}$, which implies that the diagonal entries of A must be purely imaginary.

   (b) Since $(A - A^H)^H = A^H - (A^H)^H = A^H - A = -(A - A^H)$ we see that $A - A^H$ is skew- hermitian. Since $(A + A^H)^H = A^H + (A^H)^H = A^H + A = A + A^H$, we see that $A + A^H$ is hermitian.

   (c) Since $A = (A + A^H)/2 + (A - A^H)/2$ we see that A is the sum of a hermitian matrix and a skew-hermitian matrix. If $A = B + C$ with B hermitian and C skew hermitian, we have $A^H = B - C$ so that $B = (A + A^H)/2$ and $C = (A - A^H)/2$.

3. Since A^{-1} and $A+B$ are invertible so is $(A+B)A^{-1} = I + BA^{-1}$. Since B^{-1} is invertible so is $B^{-1}(A+B)A^{-1} = B^{-1} + A^{-1}$ and $(A^{-1} + B^{-1})^{-1} = A(A+B)^{-1}B$ using the fact that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

4. (a) If A, B are $n \times n$ and $< A >_{ij} = a_{ij}$, $< B >_{ij} = b_{ij}$ then

$$\operatorname{tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{i,j} a_{ij}bji = \sum_{i,j} b_{ij}a_{ji} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki} = \operatorname{tr}(BA)$$

   (b) $\operatorname{tr}(S^{-1}AS = \operatorname{tr}(ASS^{-1} = \operatorname{tr}(A)$

5. We let $b^T = \begin{bmatrix}a & b & c\end{bmatrix}$ and row reduce $\begin{bmatrix}A & | & b\end{bmatrix}$ to echelon form.

$$\begin{bmatrix}4 & -1 & 2 & 6 & a\\ -1 & 5 & -1 & -3 & b\\ 3 & 4 & 1 & 3 & c\end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix}-1 & 5 & -1 & -3 & b\\ 4 & -1 & 2 & 6 & a\\ 3 & 4 & 1 & 3 & c\end{bmatrix} \begin{matrix}R_2 \to R_2 + 4R_1\\ R_3 \to R_3 + 3R_1\end{matrix} \begin{bmatrix}4 & -1 & 2 & 6 & a\\ 0 & 19 & -2 & -6 & a+4b\\ 0 & 19 & -2 & -6 & c+3b\end{bmatrix}$$

$R_3 \to R_3 - R_2 \begin{bmatrix}4 & -1 & 2 & 6 & a\\ 0 & 19 & -2 & -6 & a+4b\\ 0 & 0 & 0 & 0 & c-a-b\end{bmatrix}$ Hence $Ax = b$ is solvable iff $c = a + b$ in which case $x^T = \begin{bmatrix}(5a + b - 9s - 27t)/19 & (a + 4b + 2s + 6t)/19 & s & t\end{bmatrix}$ with s, t arbitrary.

6. (a) $\begin{bmatrix}2 & 4\\ 2 & 1\end{bmatrix} R_2 \to R_2 - R_1 \begin{bmatrix}2 & 4\\ 0 & -3\end{bmatrix}$ so that $\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 2 & 1\end{bmatrix} = \begin{bmatrix}2 & 4\\ 0 & -3\end{bmatrix}$ which gives
   $\begin{bmatrix}2 & 4\\ 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & -3\end{bmatrix} = \begin{bmatrix}2 & 0\\ 2 & -3\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$

   (b) $\begin{bmatrix}-2 & -4 & 2 & -2\\ 1 & 5 & 5 & -8\\ -1 & 0 & 7 & -11\\ 2 & 7 & 3 & -3\end{bmatrix} \begin{matrix}R_2 \to R_2 + (1/2)R_1\\ R_3 \to R_3 - (1/2)R_1\\ R_4 \to R_4 + R_1\end{matrix} \begin{bmatrix}-2 & -4 & 2 & -2\\ 0 & 3 & 6 & -9\\ 0 & 2 & 6 & -10\\ 0 & 3 & 5 & -5\end{bmatrix} \begin{matrix}R_3 \to R_3 - (2/3)R_2\\ R_4 \to R_4 - R_2\end{matrix}$
   $\begin{bmatrix}-2 & -4 & 2 & -2\\ 0 & 3 & 6 & -9\\ 0 & 0 & 2 & -4\\ 0 & 0 & -1 & 4\end{bmatrix} R_4 \to R_4 + (1/2)R_3 \begin{bmatrix}-2 & -4 & 2 & -2\\ 0 & 3 & 6 & -9\\ 0 & 0 & 2 & -4\\ 0 & 0 & 0 & 2\end{bmatrix}$ so that

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 1/2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -2/3 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 1/2 & 1 & 0 & 0\\ -1/2 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2 & -4 & 2 & -2\\ 1 & 5 & 5 & -8\\ -1 & 0 & 7 & -11\\ 2 & 7 & 3 & -3\end{bmatrix} = \begin{bmatrix}-2 & -4 & 2 & -2\\ 0 & 3 & 6 & -9\\ 0 & 0 & 2 & -4\\ 0 & 0 & 0 & 2\end{bmatrix}$$

$$\begin{bmatrix}-2 & -4 & 2 & -2\\ 1 & 5 & 5 & -8\\ -1 & 0 & 7 & -11\\ 2 & 7 & 3 & -3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ -1/2 & 1 & 0 & 0\\ 1/2 & 2/3 & 1 & 0\\ -1 & 1 & -1/2 & 1\end{bmatrix}\begin{bmatrix}-2 & -4 & 2 & -2\\ 0 & 3 & 6 & -9\\ 0 & 0 & 2 & -4\\ 0 & 0 & 0 & 2\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0 & 0\\ 1 & 3 & 0 & 0\\ -1 & 2 & 2 & 0\\ 2 & 3 & -1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & -1 & 1\\ 0 & 1 & 2 & -3\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

(c) $\begin{bmatrix}1 & 2 & -6\\ -3 & 4 & 7\\ 2 & 4 & 3\end{bmatrix} \begin{matrix}R_2 \to R_2 + 3R_1\\ R_2 \to R_2 - 2R_1\end{matrix} \begin{bmatrix}1 & 2 & -6\\ 0 & 7 & -11\\ 0 & 0 & 15\end{bmatrix}$ so that $\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -6\\ -3 & 4 & 7\\ 2 & 4 & 3\end{bmatrix} = \begin{bmatrix}1 & 2 & -6\\ 0 & 7 & -11\\ 0 & 0 & 15\end{bmatrix}$ and

$$\begin{bmatrix}1 & 2 & -6\\ -3 & 4 & 7\\ -2 & 4 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & -6\\ 0 & 7 & -11\\ 0 & 0 & 15\end{bmatrix}$$

7. (a) $\begin{bmatrix}0 & 2\\ 2 & 4\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 4\\ 0 & 2\end{bmatrix} = \begin{bmatrix}0 & 2\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$

(b) $\begin{bmatrix}2 & 6 & -4\\ -4 & -12 & 11\\ 3 & 14 & -16\end{bmatrix} \begin{matrix}R_2 \to R_2 + 2R_1\\ R_3 \to R_3 - (3/2)R_1\end{matrix} \begin{bmatrix}2 & 6 & -4\\ 0 & 0 & 3\\ 0 & 5 & -10\end{bmatrix} \; R_2 \leftrightarrow R_3 \; \begin{bmatrix}2 & 6 & -4\\ 0 & 5 & -10\\ 0 & 0 & 3\end{bmatrix}$ which gives

$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -3/2 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 6 & -4\\ -4 & -12 & 11\\ 3 & 14 & -16\end{bmatrix} = \begin{bmatrix}2 & 6 & -4\\ 0 & 5 & -10\\ 0 & 0 & 3\end{bmatrix}$ and hence

$\begin{bmatrix}1 & 0 & 0\\ -3/2 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}2 & 6 & -4\\ -4 & -12 & 11\\ 3 & 14 & -16\end{bmatrix} = \begin{bmatrix}2 & 6 & -4\\ 0 & 5 & -10\\ 0 & 0 & 3\end{bmatrix}$ which gives

$$\begin{bmatrix}2 & 6 & -4\\ -4 & -12 & 11\\ 3 & 14 & -16\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 3/2 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 6 & -4\\ 0 & 5 & -10\\ 0 & 0 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ -4 & 5 & 0\\ 3 & 0 & 3\end{bmatrix}\begin{bmatrix}1 & 3 & -2\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$$

(c) $\begin{bmatrix}2 & 4 & 0 & -2\\ -4 & -8 & 0 & 3\\ 3 & 7 & 2 & 4\\ 5 & 6 & 1 & -8\end{bmatrix} \begin{matrix}R_2 \to R_2 + 2R_1\\ R_3 \to R_3 - (3/2)R_1\\ R_4 \to R_4 - (5/2)R_1\end{matrix} \begin{bmatrix}2 & 4 & 0 & -2\\ 0 & 0 & 0 & -1\\ 0 & 1 & 2 & 7\\ 0 & -4 & 1 & -3\end{bmatrix} \; R_2 \leftrightarrow R_3 \; \begin{bmatrix}2 & 4 & 0 & -2\\ 0 & 1 & 2 & 7\\ 0 & 0 & 0 & -1\\ 0 & -4 & 1 & -3\end{bmatrix}$

$R_4 \to R_4 + 4R_2 \; \begin{bmatrix}2 & 4 & 0 & -2\\ 0 & 1 & 2 & 7\\ 0 & 0 & 0 & -1\\ 0 & 0 & 9 & 25\end{bmatrix} \; R_3 \leftrightarrow R_4 \; \begin{bmatrix}2 & 4 & 0 & -2\\ 0 & 1 & 2 & 7\\ 0 & 0 & 9 & 25\\ 0 & 0 & 0 & -1\end{bmatrix}$ which gives

$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 4 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ -3/2 & 0 & 1 & 0\\ -5/2 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & 0 & -2\\ -4 & -8 & 0 & 3\\ 3 & 7 & 2 & 4\\ 5 & 6 & 1 & -8\end{bmatrix} = \begin{bmatrix}2 & 4 & 0 & -2\\ 0 & 1 & 2 & 7\\ 0 & 0 & 9 & 25\\ 0 & 0 & 0 & -1\end{bmatrix}$ This yields

$\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 4 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ -3/2 & 1 & 0 & 0\\ -5/2 & 0 & 1 & 0\\ 2 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}2 & 4 & 0 & -2\\ -4 & -8 & 0 & 3\\ 3 & 7 & 2 & 4\\ 5 & 6 & 1 & -8\end{bmatrix} = \begin{bmatrix}2 & 4 & 0 & -2\\ 0 & 1 & 2 & 7\\ 0 & 0 & 9 & 25\\ 0 & 0 & 0 & -1\end{bmatrix}$ and hence

$$\begin{bmatrix}2 & 4 & 0 & -2\\ -4 & -8 & 0 & 3\\ 3 & 7 & 2 & 4\\ 5 & 6 & 1 & -8\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 3/2 & 1 & 0 & 0\\ 5/2 & -4 & 1 & 0\\ -2 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & 0 & -2\\ 0 & 1 & 2 & 7\\ 0 & 0 & 9 & 25\\ 0 & 0 & 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 0 & 0\\ 3 & 1 & 0 & 0\\ 5 & -4 & 9 & 0\\ -4 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & 0 & -\\ 0 & 1 & 2 & \\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & \end{bmatrix}$$

8. If V_p is the $p \times p$ Vandermonde determinant we have $V_p = (x_p - x_1)(x_p - x_2)\cdots(x_p - x_{p-1})V_{p-1}$ as can be seen by, starting with the last column and going down, replace the column by x_p times the previous column, expanding along the last row and factoring out $x_i - x_p$ from the i-th row of the resulting $(p-1) \times (p-1)$ determinant. The result then follows by induction.

9. $$\begin{vmatrix}x^2 & xy & y^2 & x & y & 1\\ .36 & 1.2 & 4 & .6 & 2 & 1\\ 1 & 2.3 & 5.29 & 1 & 2.3 & 1\\ 1.96 & 3.36 & 5.76 & 1.4 & 2.4 & 1\\ 4 & 5 & 6.25 & 2.0 & 2.5 & 1\\ 5.76 & 5.76 & 5.76 & 2.4 & 2.4 & 1\end{vmatrix} = .000880x^2 - .005440xy + .013120y^2 + .0097120x - .0512960y + .05049600 = 0.$$

10. If $y^T = [y_1, .., y_p]$ then $y^T y = y_1^2 + .. + y_p^2 = 0$ iff $y = 0$. Hence $Ax = 0 \implies x^T A^T A x = 0 \implies (Ax)^T(Ax) = 0 \implies Ax = 0 \implies x = 0$ since A is $p \times q$ and rank$(A) = q$.