McGill University
Math 270A: Applied Linear Algebra
Assignment 2: due Thursday September 30 , 1999

1. Let V, W be vector spaces over a field F. Prove that $V \times W = \{(v, w) \mid v \in V, w \in W\}$ is a vector space over F under the operations of addition and multiplcation by scalars defined by
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$$
$$c(v, w) = (cv, cw)$$

2. Let V be a real vector space. Show that $V \times V$ becomes a complex vector space if multiplication by complex scalars is defined by
$$(a + bi)(u, v) = (au - bv, av + bu).$$
Deduce that $(u, v) = (u, 0) + i(v, 0)$. This complex vector space is denoted by $V_{\mathbb{C}}$ and is called the complexification of V.

3. Which of the following subsets W of the given vector space V over the given field F is a subspace of V? Justify your assertions.
   (a) $V = \mathbb{R}^3, F = \mathbb{R}, W = \{(x, y, z) \mid x^2 + y^4 = 0\}$;
   (b) $V = \mathbb{C}^3, F = \mathbb{C}, W = \{(x, y, z) \mid x^2 + y^4 = 0\}$;
   (c) $V = \mathbb{R}^\infty, F = \mathbb{R}, W = \{x = (x_0, x_1, \ldots, x_n, \ldots) \in V \mid x_{n+2} = 5x_{n+1} + 6x_n \text{ for all } n \geq 0\}$;
   (d) $V = \mathbb{R}^{[0,1]}, F = \mathbb{R}, W = \{f \in V \mid f(x) = f(x^2) \text{ for all } x\}$;
   (e) $V = \mathbb{R}^{[0,1]}, F = \mathbb{R}, W = \{f \in V \mid f(x) = f(x)^2 \text{ for all } x\}$;

4. Which of the following sequences of vectors $v_1, v_2, \ldots, v_n$ in the given vector space over the given field F are linearly independent? Justify your assertions.
   (a) $V = \mathbb{R}^{2\times 2}, F = \mathbb{R}, \quad v_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & 5 \\ 4 & 5 \end{bmatrix}, v_4 = \begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$;
   (b) $V = \mathbb{R}[t], F = \mathbb{R}, \quad v_1 = (1 + t)^2, v_2 = (2 + t)^2, v_3 = (3 + t)^2, v_4 = (4 + t)^2$;
   (c) $V = \mathbb{R}^{\mathbb{R}}, F = \mathbb{R}, \quad v_1 = e^x, v_2 = e^{2x}, v_3 = e^{-x}$;
   (d) $V = V = \mathbb{R}^{\mathbb{R}}, f = \mathbb{R}, \quad v_1 = \sin^2(x), v_2 = \cos^2(x), v_3 = \cos(2x)$;
   (e) $V = \mathbb{C}^3, F = \mathbb{C}, \quad v_1 = (1, 2 + i, i), v_2 = (1 + i, 1, i), v_3 = (1, 0, i)$.

5. Given the matrix $A = \begin{bmatrix} 1 & 4 & 0 & 2 & 2 \\ 2 & 8 & 1 & 5 & 4 \\ 1 & 4 & 1 & 3 & 2 \\ 1 & 4 & -1 & 1 & 2 \end{bmatrix}$,
   (a) find bases for the row space and null space of A and A^T;
   (b) show that $f_1 = (-6, 1, 2, -2, 3), f_2 = (8, -3, -1, 1, 1), f_3 = (2, -2, -2, 2, 1)$ and $g_1 = (12, -3, 1, -1, 1), g_2 = (0, 0, 1, -1, 1), g_3 = (0, 1, 3, -3, 1)$ are both bases for the solution space of the system of homogeneous equations with coefficient matrix A. Find the coordinate vector of the solution $v = (2, -4, 1, -1, 8)$ with respect to each of these two bases.
   (c) find the transition matrices between the two bases given in (b). Show that these two matrices are inverses of one another.