1 The Implicit Function Theorem

Suppose that (a, b) is a point on the curve $F(x, y) = 0$ where and suppose that this equation can be solved for y as a function of x for all (x, y) sufficiently near (a, b). Then this part of the curve is the graph of a function $y = \varphi(x)$ on some interval $|x - a| < h$ with $\varphi(a) = b$. If $\varphi'(x)$ exists, we can compute it by differentiating both sides of the equation $F(x, \varphi(x)) = 0$ with respect to x to get

$$\frac{\partial F}{\partial x}(x, \varphi(x)) + \frac{\partial F}{\partial y}(x, \varphi(x))\varphi'(x) = 0$$

providing that the partial derivatives exist. If $\frac{\partial F}{\partial y}(x, \varphi(x)) \neq 0$, we can solve for $\varphi'(x)$ and obtain the well known formula

$$\varphi'(x) = -\frac{\frac{\partial F}{\partial x}(x, \varphi(x))}{\frac{\partial F}{\partial y}(x, \varphi(x))}$$

or, more classically,

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

The precise conditions under which the existence of h and φ is assured are furnished by the following theorem, which is the Implicit Function Theorem for functions of two variables.

Theorem 1. *If $F(a, b) = 0$ and $F(x, y)$ is continuously differentiable on some open disk with center (a, b) then, if $\frac{\partial F}{\partial y}(a, b) \neq 0$, there exists an $h > 0$ and a unique function $\varphi(x)$ defined for $|x - a| < h$ such that $\varphi(a) = b$ and $F(x, \varphi(x)) = 0$ for $|x - a| < h$. Moreover, on $|x - a| < h$, the function $\varphi(x)$ is continuously differentiable and*

$$\varphi'(x) = -\frac{\frac{\partial F}{\partial x}(x, \varphi(x))}{\frac{\partial F}{\partial y}(x, \varphi(x))}$$

There is a corresponding theorem for the case where $\frac{\partial F}{\partial x}(a, b) \neq 0$. In this case the curve $F(x, y)$ is the graph of a function of $x = \psi(y)$ near the point (a, b).

Example. Except for the two points $(\pm 1, 0)$, the curve $x^2 + y^2 = 1$ consists of the two continuously differentiable functions $y = \pm\sqrt{1 - x^2}$, $-1 < x < 1$. Notice that for $F(x, y) = x^2 + y^2$ we have $\frac{\partial F}{\partial y} = 2y$ which is zero when $y = 0$. The points $(\pm 1, 0)$ lie on the two branches $x = \pm\sqrt{1 - y^2}$, $-1 < y < 1$.

The general Implicit Function Theorem gives condition under which a system of equations

$$\begin{aligned} F_1(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0 \\ F_2(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0 \\ &\vdots \\ F_n(x_1, \ldots, x_m, y_1, \ldots, y_n) &= 0 \end{aligned}$$

can be solved for $y_1, \ldots, y_n$ as functions of $x_1, \ldots, x_m$, say $y_i = \varphi_i(x_1, \ldots, x_m)$. Differentiating the equation

$$F_i(x_1, \ldots, x_m, \varphi_1(x_1, \ldots, x_m), \ldots, \varphi_n(x_1, \ldots, x_m)) = 0$$

with respect to x_j we get

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_i}{\partial y_1}\frac{\partial \varphi_1}{\partial x_j} + \ldots + \frac{\partial F_i}{\partial y_n}\frac{\partial \varphi_n}{\partial x_j}.$$

Using the partial Jacobians

$$D_x F = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \frac{\partial F_2}{\partial x_1} & \cdots & \frac{\partial F_2}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{bmatrix}, \quad D_y F = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \frac{\partial F_2}{\partial x_1} & \cdots & \frac{\partial F_2}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial y_1} & \cdots & \frac{\partial F_n}{\partial y_n} \end{bmatrix}.$$

These equations with $1 \leq i \leq n$ can be written in the matrix form

$$D_x F + D_y F D\varphi = 0.$$

If $D_y F$ is invertible, i.e., if $|D_y F| \neq 0$, we get

$$D\varphi = -D_y F^{-1} D_x F.$$

Let $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$ and let

$$F(x, y) = (F_1(x, y), \ldots, F_n(x, y)).$$

Theorem 2 (Implicit Function Theorem). *If $F(a, b) = 0$ and $F(x, y)$ is continuously differentiable on some open disk with center (a, b) then, if $|DyF(a, b)| \neq 0$, there exists an $h > 0$ and a unique function $\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ defined for $|x - a| < h$ such that $\varphi(a) = b$ and $F(x, \varphi(x)) = 0$ for $|x - a| < h$. Moreover, on $|x - a| < h$, the function $\varphi(x)$ is continuously differentiable and*

$$D\varphi(x) = -D_y(x, \varphi(x))^{-1} D_x(x, \varphi(x)).$$

Example. Consider the equation $F(x, y) = 0$ where

$$\begin{aligned} F_1(x, y) &= x_1^2 + 2x_2 + y_1^2 + 2y_2 - 8 = 0, \\ F_2(x, y) &= x_1 - x_2^2 + y_1 - y_2^2 + 3 = 0. \end{aligned}$$

If $a = (1, 1)$, $b = (1, 2)$, we have $F(a, b) = 0$ and

$$|D_y F(a, b)| = \begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} = -10 \neq 0.$$

By the Implicit Function Theorem one has, for $x = (x_1, x_2)$ sufficiently close to $(1, 1)$,

$$y = (y_1, y_2) = (\varphi_1(x), \varphi_2(x)) = \varphi(x)$$

with $\varphi(1, 1) = (1, 2)$ and

$$D\varphi(x) = -\begin{bmatrix} 2\varphi_1(x) & 2 \\ 1 & -2\varphi_2(x) \end{bmatrix}^{-1} \begin{bmatrix} 2x_1 & 2 \\ 1 & -2\varphi_2(x) \end{bmatrix}.$$

An immediate consequence of the Implicit Function Theorem is the following theorem, known as the Inverse Function Theorem.

Theorem 3 (Inverse Function Theorem). *Let $y = f(x)$, where $y = (y_1, y_2, \cdots, y_n)$ and $x = (x_1, x_2, \cdots, x_n)$. If $DF(a)$ is invertible, then, for y near $b = f(a)$ and x near a, we have $x = g(y)$ and $Dg(b) = Df(a)^{-1}$.*