189-265A: Advanced Calculus

Assignment 2 (due Thursday October 11, 2001)

1. The centroid of a plane region R of area A has coordinates $(\overline{x}, \overline{y})$ given by

$$\overline{x} = \frac{1}{A} \iint_R x\, dxdy, \quad \overline{y} = \frac{1}{A} \iint_R y\, dxdy.$$

   (a) Show that the centroid is the unique point such that the moment of R with respect to any line passing through this point is zero. Deduce that the centroid lies on any line of symmetry of R. Recall that the moment of R with respect to the line L with equation $ax+by+c=0$ is $\iint_R h(x,y)\, dxdy$, where

$$h(x,y) = \frac{ax+by+c}{\sqrt{a^2+b^2}}$$

   is the signed distance of the point (x,y) to L.

   (b) Find the centroid of the finite region bounded by the curves $y = x^2 - x$ and $y = x$ in two ways: (a) directly and (b) by means of Green's Theorem.

2. Let C be circle $x^2 + y^2 = 2x + 2y$ with counterclockwise orientation. Compute

$$\int_C y^2 dx - x^2 dy$$

   in two ways: (i) directly and (ii) using Green's Theorem.

3. Let R the region bounded by the closed curve $x = 1 - t^2$, $y = t - t^3$, $-1 \le t \le 1$.

   (a) Find the area of R.

   (b) Find the flux of $\overrightarrow{F} = (x + \sin(y))\vec{i} + (2y - e^{x^2})\vec{j}$ out of R.

4. If C is the positively oriented boundary of the square region with vertices at $(\pm 2, 0)$, $(0, \pm 2)$, compute

$$\int_C \frac{(x-1)\, dx + y\, dy}{(x-1)^2 + y^2}.$$

5. Use the flux form of Green's Theorem to prove Green's first identity:

$$\iint_R f\nabla^2 g\, dxdy = \int_C f(\nabla g) \cdot \overrightarrow{N}\, ds - \iint_R \nabla f \cdot \nabla g\, dxdy,$$

where C is the positively oriented boundary of the plane region R and the appropriate derivatives of f and g are continuous on R.