1 Stokes' Theorem

The object of this section is to prove Stokes' Theorem.

Theorem 1 (Stokes). *If S is an orientable surface with boundary C and $\overrightarrow{F}$ is a continously differentiable vector field on S, we have*

$$\int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds = \iint_S (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S}.$$

Proof. After subdividing S into parametrized patches we are reduced to proving the theorem for the case of a parametrized surface. Let $\phi : D \mapsto \mathbb{R}^3$ denote a parametrized surface, where D is a region in the (u,v)-plane bounded by a curve C_0 for which Green's Theorem is valid. We assume that the orientation of D is positive i.e. the outward normal to D along C_0 is to the right hand relative to the direction of travel. The image of D under ϕ is a surface S bounded by the curve $C = \phi(C_0)$. We want to calculate

$$\int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds = \int_C F_1 dx + F_2 dy + F_3 dz,$$

where $\overrightarrow{F} = (F_1, F_2, F_3)$ and the functions F_i are functions of (x, y, z).

Since the curves C_0 and C are related it is natural to ask if we can find a vector field $\overrightarrow{G}$ on D such that

$$\int_C \overrightarrow{F} \cdot \overrightarrow{T}\, ds = \int_{C_0} \overrightarrow{G} \cdot \overrightarrow{T}\, ds.$$

Let C_0 be given by $\overrightarrow{w}(t) = (u(t), v(t)),\ a \leq t \leq b$. Then C is given by

$$\overrightarrow{x}(t) = \phi(\overrightarrow{w}(t)) = (x(t), y(t), z(t)).$$

We have

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = D\phi(\overrightarrow{w}(t)) \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix},$$

where $x_u = \frac{\partial x}{\partial u}$ etc. The line integral

$$\int_C F_1 dx + F_2 dy + F_3 dz = \int_a^b \overrightarrow{F}(\overrightarrow{x}(t)) \cdot \frac{d\vec{x}}{dt}\, dt.$$

The integrand $\overrightarrow{F} \cdot \frac{d\vec{x}}{dt}$ can be written in matrix form as

$$\begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} P & Q \end{bmatrix},$$

where

$$P(u, v) = F_1(\phi(u, v)) x_u(u, v) + F_2(\phi(u, v)) y_u(u, v) + F_3(\phi(u, v)) z_u(u, v),$$
$$Q(u, v) = F_1(\phi(u, v)) x_v(u, v) + F_2(\phi(u, v)) y_v(u, v) + F_3(\phi(u, v)) z_v(u, v).$$

It is immediately clear, with this definition of P and Q, that if $\overrightarrow{G} = (P, Q)$, then

$$\int_{C_0} P du + Q dv = \int_C F_1 dx + F_2 dy + F_3 dz.$$

Now Green's Theorem holds for D, i.e.,

$$\int_{C_0} P\, du + Q\, dv = \iint_D (\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})\, dudv.$$

Computing $\frac{\partial Q}{\partial u}$ and $\frac{\partial P}{\partial v}$, we get

$$\begin{aligned}
\frac{\partial Q}{\partial u} =& \big\{ \frac{\partial F_1}{\partial x}\, x_u + \frac{\partial F_1}{\partial y}\, y_u + \frac{\partial F_1}{\partial z} z_u \big\}\, x_v + F_1\, x_{uv} + \\
& \big\{ \frac{\partial F_2}{\partial x}\, x_u + \frac{\partial F_2}{\partial y}\, y_u + \frac{\partial F_2}{\partial z} z_u \big\}\, y_v + F_2\, y_{uv} + \\
& \big\{ \frac{\partial F_3}{\partial x}\, x_u + \frac{\partial F_3}{\partial y}\, y_u + \frac{\partial F_3}{\partial z} z_u \big\}\, z_v + F_3\, z_{uv}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial P}{\partial v} =& \big\{ \frac{\partial F_1}{\partial x}\, x_v + \frac{\partial F_1}{\partial y}\, y_v + \frac{\partial F_1}{\partial z} z_v \big\}\, x_u + F_1\, x_{uv} + \\
& \big\{ \frac{\partial F_2}{\partial x}\, x_v + \frac{\partial F_2}{\partial y}\, y_v + \frac{\partial F_2}{\partial z} z_v \big\}\, y_u + F_2\, y_{uv} + \\
& \big\{ \frac{\partial F_3}{\partial x}\, x_v + \frac{\partial F_3}{\partial y}\, y_v + \frac{\partial F_3}{\partial z} z_v \big\}\, z_u + F_3\, z_{uv}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} &= -\frac{\partial F_1}{\partial y} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} - \frac{\partial F_1}{\partial z} \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + \frac{\partial F_2}{\partial x} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \\
&\quad - \frac{\partial F_2}{\partial z} \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} + \frac{\partial F_3}{\partial x} \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + \frac{\partial F_3}{\partial y} \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} \\
&= \big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \big) \begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix} + \big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \big) \begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix} + \big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \big) \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \\
&= \big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \big) \cdot \big(\begin{vmatrix} y_u & z_u \\ y_v & z_v \end{vmatrix}, -\begin{vmatrix} x_u & z_u \\ x_v & z_v \end{vmatrix}, \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \big) \\
&= (\nabla \times \overrightarrow{F}) \cdot \big(\frac{\partial \overrightarrow{x}}{\partial u} \times \frac{\partial \overrightarrow{x}}{\partial v} \big).
\end{aligned}$$

Therefore

$$\iint_D \big(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \big) du dv = \iint_S (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S}.$$

$\square$