Sample Final Examination

1. For each of the following series find (i) the radius of convergence and (ii) what happens at the endpoints of the interval of convergence.

$$(a)\ \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{n4^n}, \qquad (b)\ \sum_{n=0}^{\infty} \frac{x^{3n}}{64^n \cdot \sqrt{n+1}}.$$

2. (a) Find the Maclaurin series of

$$F(x) = \int_0^x e^{-t^2/2}\, dt.$$

and evaluate $F(0.1)$ correct to 5 decimal places.

(b) Without using l'Hǒpital's Rule, compute

$$\lim_{x\to 0} \frac{(e^{2x}-1)^2}{\ln(1+x)-x}$$

3. Let $g(x,y) = \frac{x^2y}{x^2+y^2}$, $h(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $g(0,0) = h(0,0) = 0$. Show that g is continuous at $(0,0)$ while h is not.

4. (a) Re-parametrize the curve

$$\mathbf{r}(t) = (2t, \cos t, \sin t)$$

in terms of arc length measured from the point where $t = 0$.

(b) For the curve in (a), find the unit tangent, unit principal normal and binormal vectors **T**,**N**,**B** as well as the curvature at any point on the curve.

5. (a) Find the equation of the tangent plane and normal line to the surface $z = 3xe^y - x^3 - e^{3y}$ at the point $(0, 0, -1)$.

(b) Find the equation of the tangent line to the curve of intersection of the two surfaces

$$2x^2 + 3yz + z^2 = 6, \quad x^2 + xy + xz = 3$$

at the point $(1, 1, 1)$.

6. Let $r = g(x,y)$, $s = h(x,y)$ be defined implicitly by $x = r^3 - s$, $y = s^3 - r$ with $g(0,0) = h(0,0) = 1$. Find

$$\frac{\partial g}{\partial x}(0,0), \quad \frac{\partial h}{\partial y}(0,0), \quad \frac{\partial^2 g}{\partial x^2}(0,0).$$

7. Suppose that $T(x,y,z) = x^3y + y^3z + z^3x$ is the temperature at the point (x,y,z) in 3-space.

(a) Calculate the directional derivative of T at the point $P(2,-1,0)$ in the direction from P to the point $Q(1,1,2)$.

(b) A mosquito is flying through space with constant speed 5 in the direction of increasing temperature. If the mosquito's direction of flight at any given point is always normal (perpendicular) to the level surface of $f(x,y,z) = 2x^2 + 3y^2 + z^2$ passing through this point, find the rate of change of temperature experienced by the mosquito when it is at the point $(2,-1,0)$. Hint: $\frac{dT}{dt} = \nabla T \cdot \vec{v}$.

8. Show that the function $f(x,y) = 3xe^y - x^3 - e^{3y}$ has a unique critical point and that this critical point is a local maximum but not a global maximum.

9. Use the Lagrange multiplier method to find the shortest distance from the origin to the curve $xy^2 = 1$.