McGill University
Math 262: Intermediate Calculus
Solutions to Written Assignment 2

1. (a) $\mathbf{r}(t) = (e^t \cos t, e^t \sin t, e^t) \implies \mathbf{r}'(t) = e^t(\cos t - \sin t, \sin t + \cos t, 1)$. Hence $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{3}e^t$. The arc length for $0 \le t \le 2$ is $\int_0^2 |\mathbf{r}'(t)|\, dt = \int_0^2 \sqrt{3}e^t\, dt = \sqrt{3}(e^2 - 1)$.

   (b) $\mathbf{T} = \mathbf{r}'(t)/|\mathbf{r}'(t)| = \frac{1}{\sqrt{3}}(\cos t - \sin t, \sin t + \cos t, 1)$, $\quad \mathbf{N} = \mathbf{T}'(t)/|\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}(-\sin t - \cos t, \cos t - \sin t, 0)$, $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{6}}(\sin t - \cos t, -\sin t - \cos t, 2)$. Now $\kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}}{dt}|/\frac{ds}{dt} = \frac{\sqrt{2}}{3}e^{-t}$ and $\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt} = -\tau\sqrt{3}e^{-t}\mathbf{N}$ while by direct computation $\frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{6}}(\sin t + \cos t, \sin t - \cos t, 0) = -\frac{1}{\sqrt{3}}\mathbf{N}$ so that $-\tau\sqrt{3}e^t = -1/\sqrt{3}$ which gives $\tau = \frac{1}{3}e^{-t}$. Hence, at $t = \pi/2$, we have $\kappa = \frac{2}{\sqrt{3}}e^{-\pi/2}$, $\quad \tau = \frac{1}{3}e^{-\pi/2}$.

2. (a) We have $\nabla T = (3x^2y + z^3, 3y^2z + x^3, 3z^2x + y^3)$ and $\mathbf{u} = \overrightarrow{PQ} = (-1, 2, 2)$ so that, at $(2, -1, 0)$, we have $D_{\mathbf{u}}T = \nabla T \cdot \mathbf{u}/|\mathbf{u}| = (-12, 8, -1) \cdot (-1, 2, 2)/3 = 26/3$.

   (b) Let $\mathbf{r}(t) = (x(t), y(t), z(t))$ be the position of the mosquito at time t. Then $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ is the velocity of the mosquito at time t. We have $|\mathbf{v}|$ = speed of mosquito =5 and the direction of $\mathbf{v}$ is, up to sign, the gradient of f at $(2, -1, 0)$, namely $(8, -6, 0)/10 = (4, -3, 0)/5$ so that, at $(2, -1, 0)$, we have $\mathbf{v} = \pm((4, -3, 0)$. At time t, the temperature of the mosquito is $T(\mathbf{r}(t))$. The rate of change of the temperature of the mosquito per unit time is therefore

   $$\frac{d}{dt}T(\mathbf{r}(t) = \nabla T(\mathbf{r}(t)) \cdot \mathbf{v}$$

   which, at the time the mosquito is at $(2, -1, 0)$, is $(-12, 8, -1) \cdot \pm(4, -3, 0) = \mp 72$. Since the mosquito is flying in the direction of increasing temperature, the rate must be positive so that $\mathbf{v} = (-4, 3, 0)$ and the rate is 72. (Things are getting hot for the mosquito!)