McGill University
Math 262: Intermediate Calculus
Solutions to Written Assignment 1

1. (a) Let $a_n = (-1)^{n-1}\frac{(x-5)^{2n}}{n\cdot 3^n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-5|^{2n+2}}{n+1\cdot 3^{n+1}}\cdot\frac{n\cdot 3^n}{|x-5|^{2n}} = \frac{|x-5|^2}{3}\cdot\frac{1}{1+1/n} \to L = |x-5|^2/3 \text{ as } n\to\infty$$

Since $L<1 \iff |x-5|<\sqrt{3}$ the ratio test says that the series converges absolutely if $|x-5|<\sqrt{3}$ and diverges if $|x-5|>\sqrt{3}$. At $x=5\pm\sqrt{3}$ the series is $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ which converges conditionally by the alternating series test. Thus the interval of convergence is $[5-\sqrt{3}, 5+\sqrt{3}]$.

(b) By the alternating series test the magnitude of the remainder after n terms is less than $|a_{n+1}|$. When $x=5.1$ we have $|a_{n+1}| = 1/10^{2n+2}3^{n+1}(n+1) < 10^{-8}$ if $n=4$. Hence 4 terms are required.

2. (a) Since $\cos(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$ and $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^n x^n/n$, we have

$$\lim_{x\to 0}\frac{\cos(2x^2)-1}{\ln(1-x^4)} = \lim_{x\to 0}\frac{-2x^4+O(x^8)}{-x^4+O(x^8)} = \lim_{x\to 0}\frac{-2+O(x^4)}{-1+O(x^4)} = 2.$$

(b) Since $\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$, we have

$$f(x) = \frac{\tan^{-1}(2(x-1)^2)}{(x-1)^2} = \frac{\sum_{n=0}^{\infty}(-1)^n 2^{2n+1}x^{4n+2}/(2n+1)}{(x-1)^2} = \sum_{n=0}^{\infty}(-1)^n 2^{2n+1}x^{4n}/(2n+1).$$

Since the coefficient of $(x-1)^8$ is $32/5$, we have $f^8(1) = 8!(32/5) = 258048$.

3. The series solution is $y = \sum_{n=0}^{\infty} a_n x^n$ with $a_0 = y(0) = 1$, $a_1 = y'(0) = 0$. Since

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n,$$

we have

$$0 = y'' + xy' + 2y = \sum_{n=0}^{\infty}((n+2)(n+1)a_{n+2} + na_n x^n + 2a_n)x^n$$

so that $(n+2)(n+1)a_{n+2} + na_n x^n + 2a_n =$ for $n\ge 0$ and hence $a_{n+2} = -a_n/(n+1)$ for $n\ge 0$. Since $a_1 = 0$ this shows that $a_n = 0$ for n odd. For $n=2k$ we have $a_n = (-1)^k/(1\cdot 3\cdots(2k-1))$, so that

$$y = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{1\cdot 3\cdots(2k-1)}$$

is the required solution. If $b_k = (-1)^k x^{2k}/(1\cdot 3\cdots(2k-1))$ we have

$$\left|\frac{b_{k+1}}{b_k}\right| = \frac{|x|^2}{2k+1} \to 0 \text{ as } k\to\infty$$

which shows that the radius of convergence of the series is ∞ by the ratio test.