McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 262

Examiner: Professor Jakobson
Associate Examiner: Professor Jaksic

Date: Thursday, December 9, 2004
Time: 14:00 A.M. - 17:00

INSTRUCTIONS

Answer all seven questions. Each question is worth 10 points. Please give a detailed explanation for each answer.
Non-programmable calculators are permitted.
This is a closed-book exam

This exam comprises the cover and two pages of questions.

Problem 1. (10 points) Consider power series

$$\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{n(n-1)}.$$

a) [2 pts] Find the radius of the convergence of the power series.

b) [2 pts] Find the interval of convergence of the power series.

c) [2 pts] Find the second derivative of the power series.

d) [4 pts] Find the sum of the original power series.

Problem 2. (10 points) Consider the circular helix

$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 4t\mathbf{k}.$$

a) [3 pts] Parametrize $\mathbf{r}$ in terms of the arc length $\mathbf{r}(s)$ measured from the point $(1, 0, 0)$ in the direction of increasing t;

b) [3 pts] Find the curvature $\kappa(t)$ and the torsion $\tau(t)$ at a general point of the curve;

c) [4 pts] Find the Frenet frame $\hat{\mathbf{T}}(t), \hat{\mathbf{B}}(t), \hat{\mathbf{N}}(t)$ at a general point of the curve.

Problem 3. (10 points) If $u = f(x, y)$ where $x = \mathrm{e}^{4t}\cos(3t)$ and $y = \mathrm{e}^{4t}\sin(3t)$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = g(s,t)\left(\frac{\partial u}{\partial s}\right)^2 + h(s,t)\left(\frac{\partial u}{\partial t}\right)^2$$

and compute $g(s, t)$ and $h(s, t)$.

Problem 4. (10 points) The function f is defined by

$$f(x, y, z) = e^x y + z\ \cos(y) + x^2 z^2.$$

The point $P = (1, 0, 2)$ lies on the surface S defined by the equation $f(x, y, z) = 6$.

a) [4 pts] Find the gradient of f at the point P, and write the equation of the tangent plane to the surface S at P.

b) [2 pts] Find the distance from the origin $(0, 0, 0)$ to the tangent plane at P.

b) [4 pts] Determine the directional derivative $D_{\mathbf{u}}f$ of the function f in the direction of the vector $\mathbf{u} = (4/5, 0, 3/5)$, and find the maximal rate of increase of f at P.

Problem 5. (10 points) Let

$$f(x,y,z) = x^2 + 3xz - 4y^2 - 2xy + 6z^2$$

and

$$g(x,y,z) = x^3 + 9y - 2z^2.$$

The surface S_1 is defined by the equation $f(x,y,z) = 20$ and the the surface S_2 is defined by the equation $g(x,y,z) = 1$. The point $P = (0,1,2)$ belongs to both surfaces. Let γ be the curve of intersection of surfaces S_1 and S_2. Determine the equation of the tangent line to the curve γ at the point P.

Problem 6. (10 points) Find all the critical points of the function $f(x,y) = xye^{x+y}$ and determine whether they are local minima, local maxima, or saddle points.

Problem 7. (10 points) Let x, y, u, v be related by

$$x\mathrm{e}^{u+v} + 2uv - 1 = 0, \qquad y\mathrm{e}^{u-v} - \frac{u}{1+v} - 2x = 0.$$

Compute partial derivatives $(\partial u/\partial x), (\partial u/\partial y), (\partial v/\partial x), (\partial v/\partial y)$ at the point where $x = 1$, $y = 2$ and $u = v = 0$.