MATH 255: Assignment 9 Solutions

1. (a) We have $\sum \dfrac{a_n}{n^q} = \sum \dfrac{a_n}{n^p} \dfrac{1}{n^{q-p}}$ and so the result follows from Dirichlet's test or Abel's test.

   (b) We have $(k-1)a_k - ka_{k+1} \geq (a-1)a_k$ for $k \geq N$. Summing from $k = n+1$ to $t = m$, we get $(a-1)\sum_{k=n+1}^{m} \leq na_{n+1} - ma_{m+1} < na_{n+1}$ which implies $r_n \leq na_{n+1}/(a-1)$. Note that $\sum a_n$ converges and $a_{n+1} < a_n$ for $n \geq N$ since $R_n > 0$ for $n \geq N$ so that $na_n \to 0$

2. $\dfrac{a_{n+1}}{a_n} = \dfrac{(n+a)(n+b)}{(n+c)(n+1)} \to 1 \implies R = 1$. At $x = 1$, we have $\dfrac{a_{n+1}}{a_n} = 1 - \dfrac{(c+1-a-b)n + c - ab}{n^2 + (c+1)n + 1}$ which implies $R_n = n(1 - a_{n+1}/a_n) \to c + 1 - a - b$. By the Gauss test we have absolute convergence for $c > a + b$ and divergence for $c \leq a + b$. If $x = -1$ we have an eventually alternating series $\sum a_n$ with $\dfrac{|a_{n+1}|}{|a_n|} = 1 - \dfrac{(c+1-a-b)n + c - ab}{n^2 + (c+1)n + 1}$. We thus have absolute convergence if $c > a + b$. If $c < a + b - 1$ we have $|a_{n+1}| \geq |a_n|$ for $n \geq N$ so that the series diverges. If $c > a + b - 1$, we have $|a_{n+1}| < |a_n|$ and $|a_{n+1}|/|a_n| \leq (1 - h/n)$ for $n \geq N$ for some $h > 0$ so that $\log|a_{n+1}| - log|a_n| \leq \log(1 - h/n) < -h/n$ for $n \geq N$. It follows that $\log|a_{n+1}| - \log|a_N| < -h\sum_{k=N}^{n} 1/k$ so that $\log|a_n| \to -\infty$ which implies $a_n \to 0$ his gives conditional convergence at $x = -1$ when $a + b - 1 < c \leq a + b$. If $c = a + b - 1$, we have $|a_{n+1}|/|a_n| = 1 + h_n/n^2$ for $n \geq N$ with h_n bounded. Then $|a_{n+1}|/|a_N| = \prod_{k=N}^{n}(1 + h_k/k^2)$. Since $\prod_N^{\infty}(1 + h_k/k^2)$ converges, a_n cannot converge to zero so that we have divergence when $x = -1$ and $c = a + b - 1$. In summary, the interval of convergence is $[-1, 1]$ when $c > a + b$; $[-1, 1)$ when $a + b - 1 < c \leq a + b$ and $(-1, 1)$ when $c \leq a + b - 1$

3. (a) Since $|a_n| \leq Mn^c$, we have $|a_n|/n^s \leq 1/n^{s-c}$ and so we have absolute convergence for $s > c + 1$. The convergence is uniform for $s \geq c + 1 + \epsilon$ since $|\sum_{k=n}^{\infty} a_n/n^s| \leq M/n^{1+\epsilon}$ for $s \geq c + 1 + \epsilon$. Since the partial sums are continuous functions of s, this implies that the sum of the series is continuous as a function of s for $s > 1 + c + \epsilon$ for any ϵ and hence for $s > 1 + c$.

   (b) $S = (1 - \dfrac{1}{2^{s-1}})\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{2}{(2n)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} a_n$, where $a_n = 0$ if n is odd and $a_n = 2/n^s$ if n is even. This gives $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = F(s)$. Since $|\sum_{k=n}^{\infty}(-1)^{k-1}/k^s| \leq 1/n^s \leq 1/n^{\epsilon}$ for $s \geq \epsilon > 0$, we see that the series for $F(s)$ is uniformly convergent for $s \geq \epsilon$. Since the partial sums are continuous, we see that $F(s)$ is continuous.

4. By Dirichlet's test for improper integrals we have $|\int_n^{\infty} \frac{\sin x}{x^s}\, dx| \leq 2/n^s \leq 2/n^{\epsilon}$ for $s \geq \epsilon > 0$. Hence $f(s)$ is the uniform limit of the functions $\int_1^n \frac{\sin x}{x}\, dx$ on $s \geq \epsilon$. We only have to show that $f_n(s)$ is continuous for $s \geq \epsilon$. But this follows from $|f_n(s) - f_n(t)| \leq \int_1^n |x^s - x^t|\, dt$ and the fact that $|x^s - x^t| \leq K|s - t|$ on $[1, n] \times I$, where I is the interval with endpoints s, t.