1. (a) (i) $1/2^{\log n} = 1/n^{\log 2} \implies$ divergence since $\log 2 < 1$;
   (ii) $1/3^{\log n} = 1/n^{\log 3} \implies$ convergence since $\log 3 > 1$;
   (iii) $1/\log n^{\log n} = 1/n^{\log\log n} < 1/n^{\log 3}$ for $n > e^3 = 20.1 \implies$ convergence since $\log 3 = 1.1 > 1$.

   (b) (ii) $r_n < 1/(\log 3 - 1)n^{\log 3 - 1} < 1/10^3 \implies n > 4.2 \times 10^40$ would be sufficient;
   (iii) $1/n^{\log\log n} \le 1/n^p$ for $n \ge e^{e^p}$ and for the series $\sum 1/n^p$ we have $r_n < 1/(p-1)n^{p-1} < 1/10^3$ if $p > 1$. To compute the sum to within .001 using these estimates we would need $n > \max(e^{e^p}, (1000/p-1)^{1/p-1})$. For $p = 2$, we get $n > \max(1618, 1000) = 1618$. Thus $n = 1619$ terms would be sufficient. To get a better estimate, we have to reduce p. However, reducing p, increases $(1000/p-1)^{1/p-1}$. To get an optimal estimate, p would have to satisfy $e^{e^p} = (1000/p-1)^{1/p-1}$. This yields $p = 1.9692$ and $n > 1293$.

2. (a) (i) $a_{n+1}/a_n = (n+1)/2^{(n-1)e^n} < 1/55$ for $n \ge 2 \implies$ convergence;
   (ii) $a_{n+1}/a_n = 1/(1+1/n)^n \downarrow 1/e < 1 \implies$ convergence;
   (iii) $\sqrt[n]{a_n} = (2/e)(1 + n^3/2^n)^{1/n} \to 2/e < 1 \implies$ convergence.

   (b) (i) $r_n < a_n/54$ so that $r_2 < .0002$. Hence two terms suffice;
   (ii) $a_{n+1}/a_n < 1/2.2$ for $n \ge 2$ so that $r_n < n!/(2.2)n^n < .0004$ if $n \ge 8$;
   (iii) $b_n = (1 + n^3/2^n)^{1/n} \downarrow$ for $n \ge 5$ and $b_{20} < 1.0004$ so that $\sqrt[n]{a_n} < .736$ for $n \ge 20$. Since $r_{26} < .736^{27}/(1 - .736) < .001$, we see that 26 terms suffice.

3. If $a_n = (\log n)^p/n^q$, then a_n does not converge to zero if $q < 0$ or $q = 0$ and $p \ge 0$. If $q > 0$ then $a_n \downarrow$ for $n > e^{p/q}$. Applying the Cauchy condensation test to $\sum a_n$, we get $2^n a_{2n} = n^p/2^{(q-1)n}$ which implies the convergence of $\sum a_n$, and hence the absolute convergence of the given series, if and only if $q > 1$ or $q = 1$ and $p < -1$. We have conditional convergence if $0 < q < 1$, or $q = 0$, $p < 0$ or $q = 1$ and $p \ge -1$.

4. (a) (ii) $\displaystyle \frac{s}{2} = \sum_{n=1}^{\infty} \frac{2}{(2n)^2} \implies \frac{s}{2} = s - \frac{s}{2} = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n^2}$;

   (ii) $\displaystyle \frac{3s}{4} = s - \frac{s}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$

   (b)
$$s = \sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) \implies \frac{s}{2} = \sum_{n=1}^{\infty}\left(\frac{1}{4n-2} - \frac{1}{4n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{4n-1} - \frac{1}{4n}\right)$$
$$s = \sum_{n=1}^{\infty}\left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) \implies \frac{s}{2} = \sum_{n=1}^{\infty}\left(\frac{1}{8n-6} - \frac{1}{8n-4} + \frac{1}{8n-2} - \frac{1}{8n}\right)$$
$$0 = \frac{s}{2} - \frac{s}{2} = \sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{4n-1} - \frac{1}{4n} - \frac{1}{8n-6} + \frac{1}{4n-4} - \frac{1}{4n-2} + \frac{1}{8n}\right)$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{8n-6} - \frac{1}{8n-4} - \frac{1}{8n-2} - \frac{1}{8n}\right)$$
   and the required result is obtained by removing brackets. This is justified since the partial sums s_{5n+i} of the resulting series converge for $0 \le i \le 4$ as they each differ from s_{5n}, the partial sums of the grouped series, by a null sequence.

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